APOLOGY FOR THE PROOF OF THE RIEMANN HYPOTHESIS

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Abstract. An apology is an explanation or defense of actions which may otherwise be misunderstood. There are several sources of misunderstanding concerning the proof of the Riemann hypothesis. An obstacle lies in the narrow perception of the Riemann hypothesis as a mechanism for counting prime numbers. The Riemann hypothesis is significant because of its significance in mathematical analysis. The proof cannot be read as an isolated argument because of its roots in the history of mathematics. Another obstacle lies in the unexpected source of the proof of the Riemann hypothesis. The proof is made possible by events which seem at first sight to have no relevance to mathematics. Exceptional people and exceptional circumstances prepared the proof of the Riemann hypothesis.

Good writing about mathematics is difficult because the expected reader knows either too much or too little. Those with graduate experience are biased by the choice of a specialty. Those without graduate experience exist in a state of ignorance. Expository writing about mathematics needs to present the reader with a view of the subject which is convincing at several levels of knowledge. Readers without graduate experience need to be supplied with information which justifies mathematical research. Readers with graduate experience need to place their speciality within a larger perspective. These objectives are achieved by a history of mathematics as it relates to the Riemann hypothesis.

The Riemann hypothesis culminates a renewal of mathematical analysis after a millenium in which Greek analysis lay dormant in libraries. The Renaissance is stimulated by the Cartesian philosophy that problems are best solved by prior thought, as opposed to the Roman philosophy that problems are solved by immediate action. Analysis is not exclusive to mathematics since it is little else than the consistent application of thought. A common feature of effective analysis is the need for hypotheses, without which no conclusion is valid. Although analysis has striking successes, the analysis applied in mathematics surpasses other applications of analysis in the extent and consistency of its logical structure. Other applications of analysis emulate the application made in mathematics.

Mathematical analysis differs in purpose from other applications of analysis. Serious projects need to exhibit an evident purpose if they expect to receive the means required for their achievement. The value of a proposed contribution is weighed against the cost of its realization. Mathematical analysis does not admit a statement of purpose which is meaningful to those without preparation. The discovery of purpose is an historical process which perpetually diversifies itself into new channels and persistently returns to a clarification of original aims.

The earliest known applications of mathematical analysis are witnessed by architectural achievements, such as Egyptian pyramids, and by astronomical observations essential to agriculture. Mathematical analysis originates as the geometry of space with numbers as accessories in measurement. Numbers are discovered as integers from which rational numbers are constructed. An essentially different purpose is discovered for mathematical analysis when geometric objects are constructed which are not measured by rational numbers.

American readers may be pleased to learn how the attraction of irrational numbers has shaped their history. The five–pronged star which is their cultural heritage has a fascination which cannot be explained by beauty. Since beauty is akin to symmetry, the six–pronged star wins when beauty is the issue. The attraction of the five–pronged star lies in its dynamic quality which appeals for action because it is less complete. The star originates in the construction of an irrational number disturbing an eye which prefers the restfulness of rational proportions.

The distinction between constructions which terminate and those which do not assigns a purpose to mathematical analysis. The Euclidean algorithm marks the discovery of mathematical analysis as applied to infinite constructions. Discoveries of purpose in the Renaissance are illustrated in the lives of René Descartes (1596–1652), Pierre de Fermat (1601–1665), and Blaise Pascal (1623–1662).

Cartesian space reinforces the classical conception of space by the introduction of rectangular coordinates. Cartesian analysis applies the properties of real numbers to obtain the properties of geometrical figures. The success of the method justifies the Cartesian philosophy that problems are solved by thought. The contribution of Descartes to science is the discovery of orderly structure in nature which exceeds previous expectations. Of significance for the Riemann hypothesis is the conception of space as structured. The characterization of the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron as regular solids demonstrates Greek awareness of the properties of space. Descartes completes this achievement with the observation that in every case the number of faces minus the number of edges plus the number of vertices is equal to two.

The fires which ravaged the classical library of Alexandria are disastrous events in the history of mathematical analysis. Those books salvaged by Muslim scholars leave an incomplete record of Greek achievement. The theorem that every positive integer is the sum of four squares is not found in any surviving book of Diophantus. Yet the conditions stated for the representation as a sum of two squares presume a knowledge of the general representation. The mathematical contributions of Fermat are stimulated by the desire to recover and continue such classical knowledge. His problem of finding positive integers a, b, and c such that

$$a^n + b^n = c^n$$

for a positive integer n challenged subsequent generations of analysts. The infinitessimal calculus is however his most original contribution to mathematical analysis.

Although the logical skills required for mathematical analysis clearly require a special education, there is no agreement about what its content should be. Examples of a successful mathematical education are instructive for those who desire to nurture mathematical talent in themselves and in others. Pascal received an exceptional education because his mother died before he reached school age. His father personally taught him the reading and writing skills of a traditional curriculum aimed at an understanding of current political, social, and religious structures in a historical perspective. When he was twelve, he learned about the nature and purpose of a discipline called geometry. Curiosity stimulated him to attempt his own implementation of that purpose. Only then did his father supply him with Euclid's *Elements*. At that time there already existed in Paris learned societies for the presentation of scientific work. The contributions of Pascal were well received initially because they contain new arguments in support of known results and eventually because the results themselves are new. Memorable contributions are combinatorial principles which underlie the binomial theorem and the calculus of finite differences. The education of Blaise Pascal is described with loving care by his sister Jacqueline in the preface to his *Pensées*. An illuminating portrait of her by the court artist Philippe de Champaigne is preserved in the museum on the site of the ancient Abbaye de Port–Royal–des–Champs. It becomes clear that her dedication as a Jansenist nun was a major ingredient in the success of her brother's education.

A dynamical contribution of Isaac Newton (1642–1729) to mathematical analysis is to treat the origin of Cartesian coordinates as a center surrounded by the trajectories of moving particles. Momentum is introduced as a concept which permeates subsequent treatment of motion. Momentum resembles position since it lies in a space isomorphic to Cartesian space. Momentum is observable by its action on position. The motion of a particle is formulated as a voyage in time through a phase space which is composed of Cartesian space and momentum space. Implicit are mappings of phase space into itself which are defined by the motion of particles in time. An evolution of the infinitessimal calculus is required for a solution of the equations of motion. Newton applies a limiting case of the calculus of finite differences. The application to planetary motion owes its success to the understanding of Cartesian space obtained from the equations of motion.

Applications of the infinitessimal calculus are typical of research results submitted to the national scientific academies founded during the Renaissance. Christian Huygens (1629–1695) presented to the Académie des Sciences a memorable analysis of the propagation of light. The computation of the area enclosed by a circle posed a problem for analysis. John Wallis (1616–1703) presented the Royal Academy with an infinite product of rational numbers which converges to π. An infinite sum of rational numbers converging to π was discovered by Wilhelm Leibniz (1646–1716). Leibniz was a member of both academies who founded a predecessor of the Preussische Akademie der Wissenschaften. The publications of national academies were an international stimulus to mathematical analysis. In Basel Jakob Bernoulli (1654–1705) discovered infinite series whose sums are computed as products like the Wallis product for π. Johann Bernoulli (1667–1748) explored applications of the infinitessimal calculus which appear in the Leibniz computation of π.

Complex analysis originates in the observation of Abraham de Moivre (1667–1754) that the complex plane shares properties of addition and multiplication with the real line. A nonzero complex number admits an inverse. The complex plane is accepted as the domain of definition for polynomials and for the exponential function. A function of a complex variable is obtained which combines the exponential function of a real variable with the sine and cosine functions of a real variable.

The Newton interpolation polynomials in the calculus of finite differences are functions which link Diophantine analysis to Cartesian analysis. Related functions are discovered by Leonard Euler (1707–1783). The gamma function appears in 1730 as an infinite limit of Newton polynomials. The classical zeta function is introduced in 1737 by an Euler product analogous to the product for the gamma function. The expected relationship between the gamma function and the zeta function is confirmed in 1761 by the functional identity for the zeta function.

Mathematical analysis was subsidized during the Enlightenment by absolute rulers who applied the resources of emerging nations to the perceived needs of the governed. Catherine the Great in Petersburg and Frederick the Great in Potsdam followed the example of Louis XIV in Versailles in maintaining courts as centers of cultural, artistic, and scientific activity. Directions indicated by Newton and Fermat dominate mathematical analysis. The differential calculus is made rigorous by the introduction of limits. Solutions of differential equations are represented by power series. Nonconstant polynomials are shown to have zeros.

The Newtonian equations of motion are derived by Jean le Rond d'Alembert (1717–

1783) by minimizing an integral of the action of momentum on position.

A quadratic identity is used by Louis de Lagrange (1706–1783) to show that every positive integer is the sum of four squares of integers. The Fermat problem is generalized as the representation of a positive integer r by positive integers a, b, and c such that

$$a^n + b^n = rc^n$$

for a positive integer n. Solutions are obtained when n is three.

The Laplace transformation originates as a preliminary form of the Fourier transformation applied by Simon Laplace (1749–1827) to the solution of the Newtonian equations of motion. The Laplacian operator is applied in a characterization of the Newtonian potential in planetary motion.

The Euler zeta function is applied by Adrien Legendre (1752–1833) in an estimate of the number of primes with a given bound.

The Enlightenment is notable not only for the advancement of science but also for the dissemination of information. An *Encyclopédie des Sciences, des Arts, et des Métiers* supplied reliable information to critical readers. When publication of the encyclopedia became prohibitive during the French Revolution, the *Encyclopedia Britannica* was founded to meet the needs of English readers. A high quality of articles on mathematical analysis is maintained in the new encyclopedia.

Fourier analysis is the decomposition of a function, which is subject to symmetries, into elementary functions exhibiting these symmetries. An example is familiar to everyone who has observed a moving fan with a drowsy eye on a hot summer day. Patterns of two, three, and four blades appear and disappear. Observation of the fan decomposes its motion into elementary components which are only partially transmitted to the observer. The techniques of Joseph Fourier (1768–1830) make this procedure effective by a relaxation of the accepted function concept. A function need not be known prior to analysis. The function is constructed indirectly by a determination of symmetric components.

Fourier analysis introduces a fundamental change in the perception of motion in space. The Newtonian equations of motion determine the position of an isolated particle in time. The flow of heat applies to a quantity determined by approximate measurements at chosen observation points. Although heat is presumably generated by the flow of particles, the motion of particles is subordinated to the induced motion of a function of their position.

The justification of Fourier analysis assigns a purpose to mathematical analysis. Fourier analysis applies in spaces which have an additive structure similar to that of Cartesian space and whose topological properties permit an invariant integration theory. Although the real line, the complex plane, and Cartesian space have the needed properties, other spaces exist which are similar in structure and which reveal latent properties of the more familiar spaces. In application to Cartesian space the Fourier transformation is more flexible than the Laplace transformation. These transformations share a capacity for revealing symmetry properties of Cartesian space. The Laplace transformation retains its importance in Fourier analysis by its application to the flow of heat.

An application of the Fourier transformation which discovers unexpected properties of the real line is due to Denis Poisson (1781–1840). The Poisson formula states that the sum of the values of an integrable function at the integers is equal to the sum of the values of its Fourier transform at the integers when the Fourier transform is integrable and Fourier inversion applies. The Poisson formula implies the functional identity for the Euler zeta function.

Linear analysis is an aspect of mathematical analysis whose value is enhanced by its application in Fourier analysis. The functions treated by Fourier analysis belong to vector spaces and are subjected to linear transformations of which the Fourier transformation is a fundamental example. The treatment of linear transformations is difficult even in spaces of finite dimension without a determination of invariant subspaces. A major contribution of Carl Friederich Gauss (1777–1855) is a construction of invariant subspaces for linear transformations of a vector space of finite dimension over the complex numbers into the same space. An invariant subspace is constructed in every dimension which admits a subspace. The invariant subspaces obtained are nested. If r is a positive integer, the integers modulo r inherit from the integers the additive structure permitting Fourier analysis. The topology of the finite set is discrete. The canonical measure assigns to every subset the number of its elements. The functions with complex values which are defined on the integers modulo r form a vector space of dimension r which is mapped linearily into itself by the Fourier transformation. A determination of invariant subspaces is made which prepares the Dirichlet generalization of the Euler zeta function.

The polynomials of degree less than r form a vector space of dimension r to which another application of invariant subspaces is made. Gaussian quadrature evaluates a nonnegative linear functional on polynomials, which is defined by integration on the real axis, as a sum over a finite set of real numbers determined as the zeros of a polynomial of degree r. Polynomials of degree r suitable for Gaussian quadrature are constructed from the hypergeometric series, a generalization due to Euler of the Newton interpolation polynomials. Gaussian quadrature competes with prior results of Legendre as does the Gauss estimate for the number of primes with a given bound.

The representation of functions by power series is so useful as to serve effectively as a definition of a function in complex analysis. An analytic function is defined as one which is locally represented by power series. A fundamental theorem of complex analysis is due to Augustin Cauchy (1789–1859). A function $f(z)$ of z in a plane region is analytic if, and only if, the function

$$[f(z) - f(w)]/(z - w)$$

of z is continuous in the region for every element w of the region when suitably defined at w. The value at w defines the derivative at w in the sense of complex analysis. The Cauchy formula, on which the characterization depends, states that the integral of a differentiable function over a closed curve of finite length is equal to zero. The clarification of hypotheses for the Cauchy formula assigns a purpose to complex analysis.

A theorem of Camille Jordan (1838–1921) states that a simple closed curve divides the complex plane into a bounded region and an unbounded region. A necessary condition for the validity of the Cauchy formula is differentiability in the bounded region. A sufficient condition is observed by Bernhard Riemann (1826–1866). A Riemann mapping function is a function which is analytic in the unit disk and which defines an injective mapping of the disk. The Cauchy formula is valid when the bounded region is the image of the unit disk under a Riemann mapping function.

Complex analysis is stimulated by analogies with Newtonian analysis of Cartesian space. The simple closed curves of complex analysis suggest the paths of particles in phase space. Plane regions resemble the regions generated by the motion of particles in phase space. Motion in phase space according to Alembert minimizes an integral of the action of momentum on position. An analogous principle in complex analysis is formulated by Lejeune Dirichlet (1805–1859). Riemann mappings minimize a Dirichlet integral.

Riemann expected a construction of Riemann mapping functions for all regions bounded by simple closed curves. The conjecture is correct, but proof from the Dirichlet principle fails. The verification of the conjecture by Hermann Schwarz (1843–1921) applies an estimation theory for functions analytic and bounded by one in the unit disk. A related interpolation theory for functions analytic and bounded by one in the unit disk is obtained by Léopold Fejér (1880–1959) and Fréderic Riesz (1880–1956) as an application of the ubiquitous Schwarz lemma. The definitive formulation of Issai Schur (1875–1942) prepares applications of complex analysis to the construction of invariant subspaces.

The construction of Riemann mapping functions is as fundamental to complex analysis as the Cauchy formula. An estimation theory for Riemann mapping functions originates with Ludwig Bieberbach (1886–1982). The Bieberbach conjecture states that the coefficients of a Riemann mapping function

$$c_0 + c_1 z + c_2 z^2 + \dots$$

satisfy the estimate

$$|c_n| \leq n$$

for every nonnegative integer n if the estimate applies when n is zero and when n is one. The Bieberbach proof of the conjecture for the second coefficient is sufficient for an elementary construction of Riemann mapping functions. The proof of the Bieberbach conjecture for the third coefficient given by Karl Löwner (1893–1968) is motivated by the Newtonian equations of motion in Cartesian space. Riemann mapping functions are constructed from the paths of particles moving out from the origin in the plane. The area theorem is a property of motion, analogous to the Alembert principle, which is applied in the proof of the Bieberbach conjecture for arbitrary coefficients. The proof obtained in 1984 by the author of the present apology applies Hilbert spaces of functions analytic in the unit disk constructed from the area theorem by Helmut Grunsky (1904–1986).

Another purpose to mathematical analysis is supplied by the Lagrange and Fermat problems. Cyclotomic numbers are a generalization of rational numbers which appear when factorization is applied to the solution of these problems. Cyclotomic numbers are obtained from the rational numbers by adjoining a primitive r–th root of unity for some positive integer r other than one or two. These numbers are linear combinations of powers of the given root of unity with rational coefficients. The addition and multiplication of cyclotomic numbers satisfies the axioms for a field. Linear combinations with integer coefficients are the integral elements of the field. The cyclotomic field obtained by adjoining a fourth root of unity is applied by Gauss in Fourier analysis on the integers modulo r for a positive integer r. A generalization of the Euclidean algorithm applies to Gaussian integers. An Euclidean algorithm for the cyclotomic field obtained by adjoining a cube root of unity permits a solution of the Lagrange problem when n is three. The observation that a cyclotomic field need not satisfy an Euclidean algorithm is due to Ernst Kummer (1810–1893), who introduces a fundamental concept for factorization in the ring of integral elements of a field.

An ideal is a set of integral elements which contains the origin, which contains the sum of any two elements, and which contains the product of any element with an integral element of the field. An example of an ideal is the set of integral elements which are divisible by a given nonzero integral element. Kummer shows that ideals need not be so generated.

Integral elements of a ring are considered equivalent with respect to an ideal if their difference belongs to the ideal. The properties of an equivalence relation are satisfied. Every integral element is equivalent to itself. If a is equivalent to b, then b is equivalent to a. If a is equivalent to b and b is equivalent to c, then a is equivalent to c. The set of integral elements is the union of disjoint equivalence classes. The projection into the quotient space modulo the ideal takes every integral element into the equivalence class to which it belongs. Addition and multiplication of elements of the quotient space are defined so that the projection onto the quotient space is a homomorphism. The origin of the quotient space is the image of the origin of the field. The kernel of the homomorphism, which is the set of integral elements which are mapped into the origin, is the initial ideal.

The determination of structure for quotient spaces is the contribution to mathematical analysis of Evariste Galois (1811–1832). The quotient space is a finite ring when the ideal contains a nonzero element and contains a unit when the ideal does not contain every integral element of the field. The ring is isomorphic to a finite product of rings in which the noninvertible elements form an ideal. When the ring is a field, the number of its elements is a power of a prime. A unique field is constructed which has a given prime power as the number of its elements.

The Euler zeta function is constructed in Fourier analysis on the real line by Carl Jacobi (1804–1851). The construction is an application of the Laplace transformation for the line as it appears in the treatment of heat flow in the plane by Fourier. A compactification of the line is implicit since the flow of heat is confined to a horizontal strip of width one. The theta function is a sum of translates of the Laplace kernel for the line which produces a function periodic of period one. The theta function permits a treatment of heat flow in the strip which adapts the treatment of Fourier in the plane. The Poisson summation formula implies a functional identity for the theta function which has no analogue for the Laplace kernel in the plane. The Euler zeta function and its functional identity are derived from the Jacobi theta function and its functional identity by the Mellin transformation, obtained by change of variables from the Fourier transformation for the line.

A new interpretation of hypergeometric series is required for the properties of the theta function. Hypergeometric series are treated by Gauss as formal power series which are solutions of differential equations of second order with quadratic coefficients. The coefficients of hypergeometric series satisfy recurrence relations familiar from the binomial formula. Special functions appearing in Fourier analysis on the real line or the complex plane are expressible in hypergeometric series. In applications made by Jacobi the hypergeometric series is treated by the represented function. Hypergeometric functions have ambiguous values since analytic continuation is applied for their definition.

Dirichlet zeta functions are constructed from Fourier analysis on the complex plane as a generalization of the construction made by Jacobi for the Euler zeta function. The Gauss determination of invariant subspaces for the Fourier transformation on the integers modulo r prepares the concept of a character modulo r. A Dirichlet theta function is a sum of translates of the Laplace kernel for the line which produces a function periodic of period r determined by a character modulo r. The methods of Fourier are applied to the flow of heat in a horizontal strip of width r. The Poisson summation formula implies a functional identity for a Dirichlet theta function. Dirichlet zeta functions are obtained by the Mellin transformation from Dirichlet theta functions. A Dirichlet zeta function admits an Euler product and functional identity similar to the Euler product and functional identity for the Euler zeta function. Dirichlet zeta functions apply to Fourier analysis on cyclotomic fields and supply estimates of the number of primes in an arithmetic progression which have a given bound.

The Riemann hypothesis is a conjecture about the zeros of the Euler zeta function which permits an application of the Cauchy formula to the counting of primes with a given bound. The estimates due to Legendre and Gauss are deficient without an estimate of error. The Euler product denies zeros in a half–plane of convergence. Zeros are denied in a symmetric half–plane by the functional identity. A critical strip remains in which no information about zeros is obtained. The critical line divides the critical strip into symmetric halves.

The Riemann hypothesis is the conjecture that the zeros of the Euler zeta function in the critical strip lie on the critical line. Proofs that the Euler zeta function has no zeros on the boundary of the critical strip are due independently to Jacques Hadamard (1865–1963) and Charles de la Vallée–Poussin (1866–1962). The result confirms the Gauss and Legendre estimates as asymptotically correct. The Riemann hypothesis improves the accuracy of estimates.

The application of the Cauchy formula is made by Riemann to entire functions which are not polynomials. The real axis is treated as boundary of the upper half–plane. A limiting case of the Cauchy formula is applied since the upper half–plane is an unbounded region.

The application of the Cauchy formula to entire functions is clarified by Charles Hermite (1822–1909). The hypothesis of a zero–free upper half–plane for an entire function $E(z)$ is strengthened by the inequality

$$|E(z^-)| \leq |E(z)|$$

when z is in the upper half–plane. The inequality is strict when the functions $E(z)$ and

$$E^*(z) = E(z^-)^-$$

are linearly independent. A polynomial satisfies the inequality if it has no zeros in the upper half–plane. Hermite obtains a factorization for entire functions which are limits of polynomials having no zeros in the upper half–plane.

A linear functional on polynomials is said to be nonnegative if it has nonnegative values on polynomials whose values on the real axis are nonnegative numbers. A theorem of Thomas Stieltjes (1856–1894) represents a nonnegative linear functional on polynomials as a Stieltjes integral on the real line. A determination is made of all integrals which represent the linear functional. The axiomatic treatment of integration applies the Hermite theory of polynomials having a zero–free half–plane and justifies the Riemann application of the Cauchy formula.

The Stieltjes representation of nonnegative linear functionals is applied by David Hilbert (1862–1943) to the construction of invariant subspaces for continuous linear transformations of a Hilbert space into itself. An isometric transformation of a Hilbert space into itself, which is not a scalar multiple of the identity transformation, admits a closed invariant subspace, other than the smallest subspace and the largest subspace, which is also an invariant subspace for every continuous linear transformation which commutes with the given transformation. Hilbert interprets the Riemann hypothesis as the construction of a transformation to which the invariant subspace theory applies.

The Hilbert spaces in which the invariant subspace theory of isometric transformations is formulated are not immediately applicable to the Riemann hypothesis because of their distant relationship to complex analysis. Hilbert spaces whose elements are functions analytic in the upper half–plane are introduced in Fourier analysis by Godfrey Hardy (1877–1947). The Fourier transform of a function which is square integrable on the real line and which vanishes on the negative half–line is a function which admits an analytic extension to the upper half–plane. The Hardy space of functions analytic in the upper half–plane characterizes Fourier transforms. The elements of the space are the analytic functions $f(z)$ of z in the upper half–plane for which the least–upper bound

$$\|f\|^2 = \sup \int_{-\infty}^{+\infty} |f(x+iy)|^2 dx$$

taken over all positive numbers y is finite. Weighted Hardy spaces are applied in the proof of the Riemann hypothesis. An analytic weight function is a function $W(z)$ which is analytic and without zeros in the upper half–plane. Multiplication by $W(z)$ acts as an isometric transformation of the Hardy space onto a weighted Hardy space. If w is in the upper half–plane, multiplication by

$$(z - w)/(z - w^-)$$

is an isometric transformation of a weighted Hardy space into the same space. The transformation fails to have an everywhere defined isometric inverse since its range is the set of elements of the weighted Hardy space which vanish at w.

Functions analytic in the upper half–plane or in the lower half–plane are applied by Torsten Carleman (1892–1949) in the Fourier analysis of functions of a real variable whose Fourier transform is not defined by an integral. The Carleman method is applied to entire functions. The application requires estimates which are special to Fourier analysis. A theorem of Carleman states that the minimum modulus of two nonconstant entire functions cannot remain bounded in the complex plane when the functions have less than exponential growth. The proof applies a potential theory of subharmonic functions.

The transition from spaces of functions analytic in a half–plane to spaces of entire functions is made in Fourier analysis by Norbert Wiener (1894–1964). Since his prediction theory applies Fourier analysis in a time variable, functions analytic in the upper half–plane describe future time whereas functions analytic in the lower half–plane describe past time. Entire functions apply to a finite time segment. The construction of invariant subspaces by factorization of analytic functions is a technique fundamental to prediction theory. Since analytic functions with matrix values are factored, the construction of invariant subspaces prepares an existence theorem for invariant subspaces of continuous linear transformations of a Hilbert space into itself.

Hilbert spaces whose elements are entire functions are implicit in the Stieltjes integral representation of nonnegative linear functionals on polynomials. Although the Stieltjes spaces have finite dimension, spaces of infinite dimension are accessible by approximation. He would have explored the properties of spaces of infinite dimension had his career not been ended prematurely by his death from tuberculosis.

It remained for Wiener to construct the first interesting examples of Hilbert spaces of entire functions of infinite dimension. The interest of the spaces lies in properties due to the context of Fourier analysis in which they originate. The proof of the Riemann hypothesis is a search for structure in Hilbert spaces of entire functions. The Hilbert spaces of entire functions which appear in Fourier analysis are the simplest examples of spaces having the special properties applied in the proof of the Riemann hypothesis. The spaces are invariant under the shift which takes an entire function $F(z)$ into the entire function $F(z + ih)$ for a positive number h. The resulting transformation is self–adjoint and nonnegative. These properties characterize special spaces of Fourier analysis. A weakening of hypotheses is made in the proof of the Riemann hypothesis.

The transition from finite to infinite dimensional Hilbert spaces of entire functions is facilitated by an interpretation derived from the Newtonian equations of motion. The motion of a particle in one dimension is instructive as preparation for its motion in Cartesian space. A vibrating string is the classical model of motion constrained to one dimension. The Hilbert spaces of entire functions which appear in the Stieltjes integral representation of nonnegative linear functionals on polynomials describe dynamical systems which include vibrating strings. When the string model is accepted, the string consists of a sequence of masses held together by springs. The transition to Hilbert spaces of entire functions of infinite dimensions permits a uniform distribution of masses. A structural analysis of Hilbert spaces of entire functions results which is made by Mark Krein (1907–1989).

The structure of Hilbert spaces of entire functions can be described without reference to dynamical systems. Hilbert spaces of entire functions appear in totally ordered families. The typical structure is illustrated by the Hilbert spaces of entire functions appearing in the Stieltjes integral representation of a nonnegative linear functional on polynomials. Each Hilbert space has finite dimension r for a positive integer r and consists of the polynomials of degree less than r. Any two Hilbert spaces appearing are comparable in the sense that one is contained isometrically in the other. Since the scalar product of a Hilbert space is required to be nondegenerate, a greatest positive integer r may exit for which an associated Hilbert space of entire functions exists. If a Hilbert space exists for some positive integer, then it exists for every smaller positive integer.

A similar chain of Hilbert spaces of entire functions is constructed to include any given Hilbert space of entire functions. The chain of spaces is in general continuous. The entire functions which belong to the spaces need not be polynomials. There need be no smallest space in the chain. The Hilbert spaces of entire functions appearing in Fourier analysis illustrate the structure of spaces of infinite dimension. Krein conjectured but did not prove the uniqueness of the chain of Hilbert spaces of entire functions contained in a given space.

The hypercomplex analysis of Rowan Hamilton (1805–1865) is a noncommutative generalization of complex analysis. Although Cartesian space resembles the complex plane in additive structure, an analogous multiplicative structure is absent. Two kinds of multiplication appear when elements of the space are treated as vectors issuing from the origin. The scalar product of two vectors is a real number. The vector product of two vectors is a vector. Hamilton introduces a space of quaternions

$$t + ix + jy + kz$$

whose coordinates x, y, z, and t are real numbers. The conjugate quaternion is

$$t - ix - jy - kz.$$

A self–conjugate quaternion is a real number. A skew–conjugate quaternion is an element of Cartesian space. Multiplication by a self–conjugate quaternion is a multiplication of coordinates by the real number. Multiplication of skew–conjugate quaternions applies the vector product of Cartesian vectors. Multiplication of quaternions is associative but not commutative. It admits conjugation as an anti–automorphism. A nonzero quaternion has an inverse.

A clarification of the number concept is due to Richard Dedekind (1831–1916). Since his construction of real numbers is made from the integers, the clarification begins with the integers. The treatment assumes an axiomatic concept of sets and of transformations of sets into sets. A finite set is characterized by the surjective property of injective transformations of the set into itself. The existence of sets which are not finite is recognized as a hypothesis of mathematical analysis. The set of nonnegative integers is a generating example of an infinite set. The counting transformation, which takes each nonnegative integer into its successor, is an injective transformation of the set into itself which is not surjective since the origin is not the successor of a nonnegative integer. The expected properties of nonnegative integers are derived from a systematic application of the properties of the counting transformation by a process called induction. Invariance is a concept implicit in the formulation of induction. A set of nonnegative integers is said to be invariant under the counting transformation if it contains the successor to every nonnegative integer which it contains. Induction states that an invariant set contains all nonnegative integers if it contains the origin.

Another formulation of induction is indicated by the observation of Georg Cantor (1845–1918) that a set has more subsets than it has elements. The class of all subsets of a set is accepted as a set whose elements are the subsets of the given set. No transformation of a set into the class of its subsets is surjective. If a transformation J maps a set $\mathcal{S}$ into the class of all subsets of $\mathcal{S}$, then a subset $\mathcal{S}_\infty$ of $\mathcal{S}$ is constructed which is not equal to Js for an element s of $\mathcal{S}$. The construction of the set is an application of invariance. An element s of $\mathcal{S}$ belongs to $\mathcal{S}_\infty$ if no elements s_n of $\mathcal{S}$ can be constructed for all nonnegative integers n such that s_0 is equal to s and such that s_n belongs to Js_{n-1} when n is positive. An element s of $\mathcal{S}$ belongs to $\mathcal{S}_\infty$ if, and only if, Ts is contained in $\mathcal{S}_\infty$.

The cardinality of a set is a concept of size which is adapted to uncountable sets. The cardinality of set A is said to be less than or equal to the cardinality of set B if an injective transformation exists of A into B. If the cardinality of A is less than or equal to the cardinality of B and if the cardinality of B is less than or equal to the cardinality of A, then an injective and surjective transformation of A to B exists. Sets A and B are said to have equal cardinality.

Effective analysis requires that arbitrary sets A and B are comparable in cardinality. Either the cardinality of A is less than or equal to the cardinality of B or the cardinality of B is less than or equal to the cardinality of A. This evident property of countable sets is accepted as a hypothesis of mathematical analysis. The condition is formulated in an equivalent way as the axiom of choice. If a transformation T of set A into set B is surjective, then a transformation S of B into A exists such that the composition TS is the identity transformation on B. For every element b of B the transformation selects an element

$$a = Sb$$

of A such that

$$b = Ta.$$

Although the axiom of choice is appealing in simplicity, an equivalent formulation is preferred in applications. A set S is said to be partially ordered if the assertion a is less than or equal to b is meaningful for some elements a and b and has these properties: Every element of the set is less than or equal to itself. Element a is less than or equal to element c if a is less than or equal to b and b is less than or equal to c for some element b. Elements a and b are equal if a is less than or equal to b and b is less than or equal to a. A partially ordered set S is said to be well–ordered if every nonempty subset contains a least element. The set of nonnegative integers is well–ordered in the ordering defined by the counting transformation. The inequality a less than or equal to b for nonnegative integers a and b means that every invariant set of nonnegative integers which contains a contains b.

The Kuratowski–Zorn lemma is an equivalent formulation of the axiom of choice which is preferred in applications. A maximal element of a partially ordered set is an element for which no greater element exists. A partially ordered set contains a maximal element if an upper bound exists for every subset whose inherited partial ordering is a well–ordering. The proof of the Kuratowski–Zorn lemma from the axiom of choice constructs a well–ordered subset whose only upper bound lies in the subset.

The axiom of choice is applied to choose an upper bound $u(C)$ which lies outside of C whenever C is a well–ordered subset such that the set C' of upper bounds not in C is nonempty. A set A is defined whose elements are pairs (C, s) consisting of a well–ordered subset C and an element s of C' when C' is nonempty. A set B is defined whose elements are pairs (C, C') for a well–ordered subset C when C' is nonempty. The transformation T of A into B takes (C, s) into (C, C'). Since the transformation is surjective, a transformation S of B into A exists such that TS is the identity transformation on B. The desired element $u(C)$ of C' is defined for a well–ordered set C as the element s of C' such that S takes (C, C') into (C, s).

The proof of the Kuratowski–Zorn lemma is a construction of well–ordered subsets which applies the chosen upper bounds. A well–ordered subset C is said to be a ladder if every element s of C is the chosen upper bound for the well–ordered set of elements of C which are less than s. Since the empty set is well–ordered, a ladder exists. If A and B are ladders, then the intersection of A and B is a ladder which is either equal to A or equal to B. The union of all ladders is a ladder which has no upper bound outside itself.

The Dedekind construction of the real numbers from the rational numbers prepares an application of the Kuratowski–Zorn lemma. Convexity is an underlying concept of the construction. A set of rational numbers is said to be convex if it contains

$$a(1 - t) + bt$$

whenever it contains a and b if t is a nonnegative rational number such that $1 - t$ is nonnegative. The closure of a nonempty convex set B of rational numbers is the set B^- of rational numbers a such that the set whose elements are a and the elements of B is convex. The empty set is a convex set whose closure is defined to be itself. The closure of a convex set of rational numbers is a convex set of rational numbers whose closure is itself. A convex set of rational numbers is said to be open if it is disjoint from the closure of every disjoint convex set. The intersection of two open convex sets is an open convex set. A set of rational numbers is said to be open if it is a union of open convex sets. A set of rational numbers is said to be closed if its complement is open. A convex set is closed if, and only if, its closure is equal to itself.

The Dedekind construction of the real numbers is based on an evident property of open convex sets of rational numbers. If a nonempty open convex set A is disjoint from a nonempty convex set B, then A is contained in an open convex set which is disjoint from B and whose complement is convex. A real number which is not rational is determined by every nonempty open convex set whose complement is a nonempty open convex set. A similar construction of open sets and closed sets is made in any space in which convexity is meaningful. The Kuratowski–Zorn lemma is applied to prove the existence of open convex sets whose complement is convex. A formulation of the Hahn–Banach theorem due to Marshall Stone (1903–1989) states that a nonempty open convex set A which is disjoint from a nonempty convex set B is contained in an open convex set which is disjoint from B and whose complement is convex.

Topology is an underlying concept of the Riemann hypothesis. Topology is encountered at the most elementary level in the Dedekind construction of real numbers from rational numbers. The topology of the real line is the structure given to it by its open subsets or equivalently by its closed subsets. This structure, which is derived from convexity, facilitates the transition from numbers which have a clear construction from integers to numbers which defy a comparable description in finite terms. Topology elucidates properties which are essential to mathematical analysis.

An axiomatization of topology is due to Felix Hausdorff (1868–1942). A topology is defined on a set by prescribing a class of open subsets or equivalently a class of closed sets which are the complements of open sets. Unions of open sets are assumed to be open and intersections of closed sets are assumed to be closed. Finite intersections of open sets are assumed to be open and finite unions of closed sets are assumed to be closed. A Hausdorff space is a set, for which open and closed sets are defined, such that distinct elements are contained in disjoint open sets.

The Hausdorff axiomatization of topology is a discovery of structure for mathematical analysis. The Dedekind topology of the rational numbers, which underlies the construction of the real numbers, has a good relationship to the additive structure of the rational numbers. This property of the topology permits the real numbers to acquire an additive structure. The relationship of topology to additive structure is expressed in the continuity of addition, a transformation which takes pairs (a, b) of rational numbers into rational numbers $a + b$. The Cartesian product of the set of rational numbers with itself, which is the set of pairs (a, b) of rational numbers, acquires a topology from the Dedekind topology of the rational numbers. The topology of the Cartesian product space is defined using the coordinate projections (a, b) into a and (a, b) into b of the Cartesian product space onto the rational numbers. An open subset of the Cartesian product space is defined as a union of basic open subsets. A basic open subset of the Cartesian product space is defined by open subsets U and V of the set of rational numbers and consists of the pairs (a, b) such that a belongs to U and b belongs to V. The Cartesian product space is a Hausdorff space in the Cartesian product topology. Continuity of addition means that for every open subset U of the set of rational numbers, the set of pairs (a, b) of the Cartesian product space such that $a + b$ belongs to U is open.

A first encounter with topology creates the impression that excessive effort is expended for small gain. The value of an axiomatic treatment of what is already known is that it permits an understanding of what was not previously known. An unexpected structure which satisfies the axioms must be interesting if the original structure was interesting and if the axioms have merit. In this way Hausdorff spaces are discovered whose topology is unrelated to convexity. These spaces are interesting because they supplement the information obtained from convexity.

Examples of Hausdorff spaces which are unrelated to convexity are obtained when the space admits a sufficiently large class of sets which are both open and closed. The condition is that every open set is a union of sets which are both open and closed and every closed set is an intersection of sets which are both open and closed. Such a set is conveniently described by its characteristic function, a function which has value one on the set and which has value zero elsewhere. The properties of sets which are both open and closed are revealed by treating the characteristic function as having values in the integers modulo two. The unique Hausdorff topology of the integers modulo two is the discrete topology, for which every subset is both open and closed. The characteristic functions of sets which are both open and closed are the continuous functions with values in the integers modulo two.

These functions form an algebra over the field of integers modulo two. The addition of the characteristic functions of sets A and B is the characteristic function of the set whose elements belong to the union of A and B but not to their intersection. The multiplication of the characteristic functions of sets A and B is the characteristic function of the intersection of A and B. The function which is identically zero is the characteristic function of the empty set. The function which is identically one is the characteristic function of the full space.

The rational numbers admit topologies, other than the Dedekind topology, which are compatible with additive structure. These topologies are initially defined on the integers but extend to the rational numbers because they are compatible with multiplicative structure. The construction of topologies applies a determination of ideals of integers resulting from the Euclidean algorithm.

An example of an ideal is constructed from a positive integer r as the set of integers which are divisible by r. The quotient space of integers modulo r contains r elements which are represented by the nonnegative integers less than r. The addition and multiplication of integers modulo r resembles the addition and multiplication of integers when these representatives of equivalence classes are chosen. Integers which are divisible by r are discarded so as to maintain the same representatives in equivalence classes. An ideal of integers which contains a nonzero element contains a least positive element. If b is the least positive element of the ideal and if a is an integer, then an integer c exists such that

$$a - bc$$

is a nonnegative integer less than b. When a belongs to the ideal,

$$a - bc = 0$$

since $a - bc$ is an element of the ideal. The ideal is the set of integers which are divisible by b.

A topology for the integers results from the computation of the ideals of integers. The quotient space of the integers modulo a nontrivial ideal is a finite set which inherits addition and multiplication from the integers. A finite set admits a unique topology, the discrete topology, with respect to which it is a Hausdorff space. Every subset is both open and closed with respect to this discrete topology. Addition and multiplication are continuous as transformations of the Cartesian product of the set with itself into the set.

The adic topology of the integers is defined by the requirement of continuity of the projection into the quotient space modulo every nontrivial ideal. An example of a set which is open and closed is constructed from a nontrivial ideal and a subset A of the quotient space modulo the ideal. The open and closed set contains the integers which project into an element of A. The class of open and closed sets is closed under finite unions and finite intersection. A set is open if it is a union of open and closed sets. A set is closed if it is an intersection of open and closed sets.

If r is a positive integer, multiplication by r is an injective transformation of integers into integers. The image of the integers under the transformation is the ideal of integers which are divisible by r. The ideal is an open and closed set for the adic topology. The transformation maps every open set for the adic topology onto an open set for the adic topology. A set of integers is open for the adic topology if the transformation maps it onto an open set for the adic topology.

The adic topology of the rational numbers is derived from the adic topology of the integers. A set A of rational numbers is open for the adic topology if for every positive integer r the set of integers which are products ra with a in A is open. A set B of rational numbers is closed for the adic topology if for every positive integer r the set of integers which are products rb with b in B is closed. The rational numbers are a Hausdorff space in the adic topology. Addition is continuous as a transformation of the Cartesian product of the space of rational numbers with itself into the space of rational numbers when the rational numbers are given the adic topology. Multiplication by a rational number is a continuous transformation of the space of rational numbers into the space of rational numbers when the rational numbers are given the adic topology. The adic line is the Cauchy completion of the rational numbers in the uniform adic topology.

Fourier analysis differs from Newtonian analysis in the manner of collecting information. The properties of space are discovered in both cases by the motion of particles in time which constructs mappings of space into itself. In Fourier analysis functions of motion are observed rather than the position of particles as in Newtonian analysis. The original application of Fourier analysis is made to the flow of heat.

Although functions are as fundamental to Newtonian analysis as they are to Fourier analysis, the functions which appear in Newtonian analysis are typically continuous. The functions encountered in the infinitessimal calculus are continuous since continuity is a consequence of differentiability. Polynomials are continuous functions as are functions represented by power series. When discontinuities occur, they are caused by boundary conditions as when a ray of light is bent at an interface between air and water. It seems possible in Newtonian analysis to isolate singularities as exceptional phenomena.

Singularities of functions are of a more earnest nature in Fourier analysis since they need not be isolated. Every singularity increases the work needed to define the integrals of Fourier analysis. The techniques of integration need to be improved so as to minimize the dependence on continuity. When this is done, it needs to be determined whether integration applies to sufficiently many functions.

Answers to these questions were given by René Baire (1874–1932) whose contribution can be treated as an achievement in cardinality. The construction of uncountable sets by Cantor inspires an appreciation of countable sets. A nonempty open set of real numbers is uncountable according to Cantor since it has the same cardinality as the class of all subsets of a countable set. Baire shows that a nonempty open set of real numbers is large in another sense. A set of real numbers is said to be dense if all real numbers belong to its closure. Baire shows that a countable intersection of dense open sets is dense.

The applications of the least infinite cardinal number to properties of sets are also applications of the same cardinal number to properties of functions. A function defined on a Hausdorff space is said to be measurable in the sense of Baire if it is a pointwise limit of a sequence of continuous functions. A subset of a Hausdorff space is said to be measurable in the sense of Baire if its characteristic function is Baire measurable. A function defined on a Hausdorff space is Baire measurable if, and only if, the inverse image of every open set is Baire measurable.

Baire measurable sets are applied by Emile Borel (1871–1956) in the construction of measures. A measure is a countably additive function of sets which are said to be measurable because of the intended application as the domain of a measure. Hypotheses are required of measurable sets for the application. A countable union of measurable sets is measurable. The complement of a measurable set is measurable. The hypotheses are satisfied by the Baire measurable subsets of a Hausdorff space. Nonnegative measures are of special importance since other measures are constructed from them. A nonnegative measure is a function of measurable sets, whose values are nonnegative numbers or infinity, such that the measure of a countable union of disjoint sets is the sum of the measures of the sets. Infinity is taken in a positive sense since it is essentially a device for stating that the domain of the measure does not contain all measurable sets. It would be tedious to distinguish between sets on which the measure is defined as a number and those on which it is not. The domain of the measure consists effectively of those measurable sets on which the value of the measure is a number. The existence of sets of finite measure is therefore a significant feature of measures.

Hausdorff spaces contain closed subsets which are accepted by Borel as Baire measurable sets of finite measure. These subsets are said to be compact in the subsequent axiomatization of topology. A Hausdorff space is said to be compact if a class of closed subsets has a nonempty intersection whenever every finite subclass has a nonempty intersection. A subset of a Hausdorff space is treated as a Hausdorff space whose open sets are the intersections of the subset with open subsets of the full space. A subset of a Hausdorff space is said to be compact if it is a compact Hausdorff space in the subspace topology. Compact subsets of Hausdorff spaces are closed. An application of the Kuratowski–Zorn lemma is made in the proof that a Cartesian product of compact Hausdorff spaces is a compact Hausdorff space.

The properties of compact sets are explored by Hausdorff in the axiomatization of topology. A Hausdorff space is said to be completely regular if the topology of the space is determined by its continuous functions. The requirement is that every open set is a union of basic open sets. A basic open set is defined by a finite number of continuous real–valued functions $f_0(s), \ldots, f_r(s)$ of s in the space and consists of the elements s which satisfy the inequalities

$$-1 < f_n(s) < 1$$

for $n = 0, \ldots, r$. A construction of continuous functions is made on a compact Hausdorff space which implies that the space is completely regular. The Baire measurable subsets of a compact Hausdorff space are acceptable for the construction of measures.

The construction of an integration theory adequate for Fourier analysis is sufficiently cumbersome as to obscure the achievements of its predecessor. A more explicit integration theory is applied by Stieltjes in the representation of nonnegative linear functions on polynomials. A nonnegative linear functional on polynomials, best treated as functions $f(z)$ of a complex variable z, is defined by a nondecreasing function $\mu(x)$ of real x as an integral

$$\int f(x)d\mu(x)$$

which is a limit of finite linear combination of values of the polynomial on the real axis with nonnegative coefficients determined by increments in μ. Stieltjes determines all nondecreasing functions which represent the linear functional on polynomials of degree r for a positive integer r. Although some nondecreasing function applies without restriction on degree, such a function need not be essentially unique.

The integration of Baire measurable functions proceeds through the construction of a nonnegative measure on Baire measurable sets of real numbers which is determined on intervals by a nondecreasing function of a real variable. The integral defined by the measure agrees on continuous functions with the Stieltjes integral. Decisive properties of the integral were discovered by Henri Lebesgue (1875–1944). The Lebesgue monotone convergence theorem states that the limit of the integrals of a sequence of functions is equal to the integral of the limit function when the sequence of values of the functions is nondecreasing everywhere on the real axis. The theorem permits the construction of complete function spaces whose value is clarified by Stefan Banach (1892–1947). The Hahn–Banach theorem completes the logical structure in support of an effective Fourier analysis.

The construction of a nonnegative measure on the Baire measurable subsets of the real line from a nondecreasing function $\mu(x)$ of real x deserves attention. The definition of the measure of an open set is clear since an open set is a countable union of disjoint convex open sets. The measure of an open interval (a, b) is

$$\mu(b) - \mu(a).$$

The measure of a half–line (a, ∞) is the least upper bound of the measures of open intervals (a, b). The measure of a half–line $(-\infty, b)$ is the least upper bound of the measures of open intervals (a, b). The measure of the line is the least upper bound of the measures of all open intervals (a, b).

The measure of a subset A of the real line, if it can be defined, is clearly less than or equal to the outer measure

$$\mu(A) = \inf \mu(U)$$

which is the greatest lower bound of the measures $\mu(U)$ of open sets U containing A. Outer measure fails to be a measure since equality need not hold in the inequality

$$\mu(A \cup B) \leq \mu(A) + \mu(B)$$

when A and B are disjoint subsets of the real line. Equality does hold when A and B are Baire measurable sets.

A subset C of the real line is said to be measurable in the sense of Carathéodory if equality always holds in the inequality

$$\mu(C) \leq \mu(C \cap A) + \mu(C \cap B)$$

whenever A and B are disjoint Baire measurable sets whose union contains C. The complement of a Carathéodory measurable set is Carathéodory measurable. A countable union of sets which are Carathéodory measurable is Carathéodory measurable. Outer measure is a measure on Carathéodory measurable sets.

The construction of measures on Carathéodory measurable sets leaves open the question whether measures can be defined on a larger class of sets. Some measures exist for which all subsets of the real line are Carathéodory measurable. An example is a measure defined by a nondecreasing function which is a step function. The set of discontinuities of a step function is countable. The outer measure of a set is then equal to the outer measure of the intersection of the set with the set of discontinuities.

The Banach measure problem concerns the existence of other nonnegative measures which are defined on all subsets of the real line and which are finite on compact sets. Since such a measure is defined on Baire measure sets, it is determined on these sets by a nondecreasing function. The measure is also determined by the nondecreasing function on Carathéodory measurable sets. The issue in the Banach measure problem is whether the measure can be extended by some possibly different procedure to the class of all subsets of the real line when the nondecreasing function is not a step function. The existing formulation of Fourier analysis is acceptable only when the answer to the question is no. If the answer is yes, then the merits of another formulation remain to be explored.

Lebesgue measure for the real line is the measure on Baire measurable sets defined by the increasing function

$$\mu(x) = x$$

of real x. The measure is essentially characterized by invariance under translation. If A is a Baire measurable subset of the real line and if h is a positive number, then a Baire measurable subset B is defined as the set of sums

$$b = a + h$$

with a in A. The Lebesgue measure of B is equal to the Lebesgue measure of A. A nonnegative measure on the Baire measurable subsets of the real line which is invariant under translation is a constant multiple of Lebesgue measure.

The Hilbert space of equivalence classes of functions which are square integrable with respect to Lebesgue measure is applied in the definition of the Fourier transformation for the real line. A Baire measurable function $f(x)$ of real x is said to be square integrable with respect to Lebesgue measure if the integral

$$\int |f(x)|^2 dx$$

is finite. Square integrable functions $f(x)$ and $g(x)$ of real x are considered equivalent if the integral

$$\int |g(x) - f(x)|^2 dx$$

is zero. The space of equivalence classes of square integrable functions with respect to Lebesgue measure is a complete metric space in which the integral is the square of the distance from f to g. Since equivalent continuous functions are equal and since the continuous functions represent a dense set of square integrable functions, the space of square integrable functions is a metric completion of the space of continuous functions which are square integrable.

The Fourier transformation for the real line is a property of additive structure. Multiplicative structure appears in a construction of homomorphisms of additive structure. If t is a real number, a homomorphism of additive structure of the real numbers into the real numbers is defined by taking x into the product tx. The homomorphism is continuous for the Dedekind topology of the real line. Every homomorphism of additive structure of the real numbers into the real numbers which is continuous for the Dedekind topology is defined by a unique real number t. Since homomorphisms can be added to produce homomorphisms, the set of homomorphisms of the real line into itself admits an additive structure. The dual space obtained is isomorphic to the real line in its additive structure. The dual space acquires a topology through its action on the real line. The topology of homomorphisms t is the weakest topology with respect to which tx is a continuous function of t for every real number x. The dual space is isomorphic in additive and topological structure to the real line. Duality validates the concept of momentum introduced by Newton. Momentum space is isomorphic to Cartesian space in additive and topological structure. The action of momentum on position produces real numbers.

The treatment of Fourier analysis for the real line presumes a knowledge of Fourier analysis for the real numbers modulo 2π. Real numbers modulo 2π are familiar as angles in the de Moivre definition of the exponential as a function of a complex variable. Functions of an angle variable are functions $f(x)$ of a real variable x which are periodic of period 2π. An example of a continuous periodic function is the function

$$\exp(2\pi\, inx)$$

of x for every integer n. The integral

$$\int \exp(2\pi inx) dx$$

with respect to Lebesgue measure over the interval 2π is equal to zero when n is nonzero. If a continuous function $f(x)$ of real x is periodic of period 2π, the Lebesgue integral

$$2\pi a_n = \int f(x) \exp(2\pi inx) dx$$

over the interval 2π defines a Fourier coefficient a_n for every integer n. The Lebesgue integral

$$\int |f(x)|^2 dx = \sum |a_n|^2$$

over the interval 2π is equal to the sum of squares of absolute values of Fourier coefficients. Fourier inversion

$$f(x) = \sum a_n \exp(-2\pi i n x)$$

applies almost everywhere when the sum is absolutely convergent.

Fourier appreciated the need for discontinuous functions which are periodic of period 2π. Arbitrary periodic functions cannot be applied since integrability is required. The desired class of functions is identified by Baire measurability, by the Borel formulation of measure, and by the Lebesgue definition of integral. The desired space consists of the equivalence classes of Baire measurable functions which are square integrable with respect to Lebesgue measure over the interval 2π.

The Fourier transform for the real line of a function $f(x)$ of real x, which is integrable with respect to Lebesgue measure, is the bounded continuous function

$$g(x) = \int f(t) \exp(2\pi i x t) dt$$

of real x defined by integration with respect to Lebesgue measure. The identity

$$\int |f(x)|^2 dx = \int |g(x)|^2 dx$$

holds with integration with respect to Lebesgue measure if the function $f(x)$ of real x is square integrable with respect to Lebesgue measure. A dense set of elements of the space of equivalence classes of square integrable functions with respect to Lebesgue measure are represented by bounded integrable functions with respect to Lebesgue measure. The Fourier transformation is extended as a transformation of equivalence classes of square integrable functions into equivalence classes of square integrable functions which maintains the identity. If a square integrable function $g(x)$ of real x is equivalent to the Fourier transform of a square integrable function $f(x)$ of real x, then the square integrable function $f(-x)$ of real x is equivalent to the Fourier transform of the square integrable function $g(x)$ of real x.

The Poisson summation formula

$$\sum f(n) = \sum g(n)$$

states that the sums over the integers n for an integrable function $f(x)$ of real x and an integrable function $g(x)$ of real x are equal if the function $g(x)$ of x is the Fourier transform of the function $f(x)$ of real x and if the function $f(-x)$ of real x is the Fourier transform of the function $g(x)$ of real x.

The Poisson summation formula belongs to a formulation of Fourier analysis in which the line is compactified by introducing a topology compatible with additive structure. The topology combines the Dedekind topology with the adic topology of the rational numbers. The mixing of topologies is made on a Cartesian product space.

The Cartesian product of the real line and the adic line is the set of pairs (c_+, c_-) consisting of a real number c_+ and an adic number c_-. The sum of elements (a_+, a_-) and

(b_+, b_-) of the Cartesian product is the element (c_+, c_-) of the Cartesian product whose real component

$$c_+ = a_+ + b_+$$

is the sum of real components and whose adic component

$$c_- = a_- + b_-$$

is the sum of adic components. The Cartesian product is given the Cartesian product topology of the real line and the adic line. An equivalence relation is defined on the Cartesian product space. Elements (a_+, a_-) and (b_+, b_-) of the Cartesian product space are considered equivalent if $b_+ - a_+$ and $a_- - b_-$ are equal rational numbers. The elements of the Cartesian product space which are equivalent to the origin form a discrete subset which is closed under addition and which contains $(-a_+, -a_-)$ whenever it contains (a_+, a_-). The quotient space inherits an additive structure and a topology with respect to which it is a compact Hausdorff space. A fundamental region is the set of elements (c_+, c_-) of the Cartesian product space with c_+ in the interval $(-\frac{1}{2}, \frac{1}{2})$ and c_- integral. The closure of the fundamental region is the set of elements (c_+, c_-) of the Cartesian product space with c_+ in the interval $[-\frac{1}{2}, \frac{1}{2}]$ and c_- integral. The fundamental region is an open subset of the Cartesian product space whose closure is compact. An element of the Cartesian product space is equivalent to an element of the closure of the fundamental region. Equivalent elements of the fundamental region are equal. Each equivalence class is a closed subset of the Cartesian product space which inherits a discrete topology.

The quotient space inherits an addition from the Cartesian product space and a multiplication by elements of the Cartesian product space whose coordinates are equal rational numbers. The projection onto the quotient space is a homomorphism of additive structure.

An isomorphism of additive structure, which takes the real line into the quotient space, is defined by taking c into $(c, 0)$ for every real number c. Since the image of the real line is dense in the quotient space, the quotient space is a compactification of the real line in a topology which is compatible with additive structure and with multiplication by rational numbers.

An isomorphism of additive structure, which takes the adic line into the quotient space, is defined by taking c into $(0, c)$ for every element c of the adic line. Since the image of the adic line is dense in the quotient space, the quotient space is a compactification of the adic line in a topology which is compatible with additive structure and with multiplication by rational numbers.

Since the rational numbers are dense in the real line and in the adic line, the quotient space is a compactification of the rational numbers in a topology which is compatible with additive structure and with multiplication by rational numbers. The compactification of the rational numbers implied by the Poisson summation formula formulates consequences of the multiplicative action of the rational numbers on the Fourier analysis of a line.

Fourier analysis is relevant to the Riemann hypothesis since the Poisson summation formula is applied in the proof of the functional identity for Euler and Dirichlet zeta functions. Fourier analysis is also relevant since the Poisson formula is applied to functions originating in the flow of heat, which is treated by the Laplace transformation. Heat flow is a different application of Fourier analysis since it is treated in a plane region. Hilbert spaces of functions analytic in the upper half–plane appear in the characterization of Laplace transforms.

The upper half–plane is a region conformally equivalent to the unit disk, a region preferred in Riemann mapping. An analytic function $f(z)$ of z in the unit disk which maps the disk injectively onto itself is a quotient

$$f(z) = (Az + B)/(Cz + D)$$

of linear functions for a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with complex entries which satisfies the identity

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A^- & B^- \\ C^- & D^- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These transformations form a group under composition, which is computable as a matrix product.

The unit disk admits an analytic structure whose automorphisms are computed by a group of matrices. A generalization of Fourier analysis results in which additive structure is lost but in which multiplicative structure is enriched by the noncommutative nature of matrix multiplication. The special functions of the hyperbolic geometry of the unit disk are hypergeometric series. The parabolic geometry of the complex plane is a limiting case of the hyperbolic geometry of the unit disk as it is of the elliptic geometry of the unit sphere. The special functions of the unit sphere are more elementary than those of the unit disk since the unit sphere is compact. The special functions of the complex plane retain the nontrivial nature of the special functions of the unit disk without loss of simplicity of the special functions of the unit sphere.

Since the unit disk is not compact, the space can be treated in the same way as the line to produce compactifications. The construction applies discrete subgroups of the group of automorphisms of the disk which were introduced by Felix Klein (1849–1925) and Henri Poincaré (1854–1912). The quotient space of the disk under the group action is a Riemann surface. The matrices which parametrize the elements of the subgroup resemble the integers when the quotient space is compact. Examples of discrete groups which are related to theta functions are best described in the hyperbolic geometry of the upper half–plane.

An analytic function $f(z)$ of z in the upper half–plane which maps the upper half–plane injectively onto itself is a quotient

$$f(z) = (Az + B)/(Cz + D)$$

of linear functions for a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with complex entries which satisfies the identity

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A^- & B^- \\ C^- & D^- \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

These transformations form a group under composition, which is computable as a matrix product. The modular group is the discrete subgroup whose elements are the matrices with integer entries and determinant one.

The modular group is generated by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

of order three and the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

of order four. A homomorphism exists of the modular group into the complex numbers of absolute value one whose value on the matrix of order three is a primitive fourth root of unity and whose value on the matrix of order four is a primitive third root of unity. The matrices on which the homomorphism has value one form a normal subgroup whose quotient group is isomorphic to the integers modulo twelve. The matrices on which the homomorphism is a fourth root of unity form a normal subgroup whose quotient group is isomorphic to the integers modulo four. The matrices on which the homomorphism is a cube root of unity form a normal subgroup whose quotient group is isomorphic to the integers modulo three.

A fundamental region for the action of the modular group is the set of elements z of the upper half–plane which lie in the strip

$$-1 < z + z^- < 1$$

and outside

$$z^- z > 1$$

of the unit circle. An element of the upper half–plane is equivalent under the modular group to an element of the closure of the region. Equivalent elements of the region are equal. The element

$$-\tfrac{1}{2} + \tfrac{1}{2}\, i\sqrt{3}$$

on the boundary is left fixed by the generator of order three. The element

$$i$$

on the boundary is left fixed by the generator of order four. Boundary elements of the fundamental region are identified by the generator of order four and by the product of

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

of the generator of order four and the generator of order three. The quotient space is homeomorphic to a punctured sphere which is compactified by the addition of a single point.

Modular forms are constructed to resemble theta functions without presuming an origin in Fourier analysis. Modular forms of order ν are defined for nonnegative integers ν. A modular form of order ν is an analytic function $F(z)$ of z in the upper half–plane which satisfies the identity

$$F(z) = \frac{1}{(Cz+D)^{1+\nu}}\; F\left(\frac{Az+B}{Cz+D}\right)$$

for every element

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of the normal subgroup of index twelve in the modular group. A modular form of order ν is a linear combination of modular forms of order ν which satisfy a related identity for every element of the modular group. A homomorphism of the modular group into the complex numbers of absolute value one is applied in the statement of the identity.

The resemblance to theta functions is heightened when the modular form is required to have a power series expansion

$$F(z) = \sum a_n \exp(2\pi i n z)$$

in the variable

$$\exp(2\pi i z).$$

The power series expansion endows the fundamental region with an analytic structure to produce a compact Riemann surface. These conditions are not easily satisfied. Eleven is the least positive integer ν for which a nontrivial modular form of order ν exists with the desired power series expansion. The modular form obtained is essentially unique since the space obtained has dimension one. A zeta function with Euler product analogous to the Euler product for Dirichlet zeta functions and for the Euler zeta function was constructed from the modular form by Srinivasa Ramanujan (1887–1920). He conjectured an estimate of coefficients which creates convergence of the Euler product in a half–plane analogous to that for the Euler and Dirichlet zeta functions. A critical strip is left in which no information concerning zeros is obtained. The critical line, which divides the critical strip in half, is a line of symmetry for zeros by the functional identity. A generalization of the Riemann hypothesis is indicated by the resemblance to the Euler zeta function.

The construction made by Ramanujan fails when the space of modular forms of order ν with the desired power series expansion has dimension greater than one. The coefficients of modular forms need not have the properties required for a zeta function with Euler product. A basis for the vector space of order ν is constructed by Erich Hecke (1877–1947) whose elements are modular forms having the required properties. The zeta function has an Euler product which indicates a generalization of the Riemann hypothesis when an estimate of coefficients is satisfied which generalizes the conjecture made by Ramanujan.

The Hecke construction of modular forms with zeta function having an Euler product deserves attention as a hint to the proof of the Riemann hypothesis. Commuting operators are constructed on spaces of modular forms of order ν. The desired modular forms are found as eigenfunctions of Hecke operators. The desired operators are constructed by taking $F(z)$ into

$$G(z) = \frac{1}{(Rz+S)^{1+\nu}} \, F\left(\frac{Pz+Q}{Rz+S}\right)$$

for a matrix

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

with integer entries and positive determinant. The function obtained satisfies the identity

$$G(z) = \frac{1}{(Cz+D)^{1+\nu}} \, G\left(\frac{Az+B}{Cz+D}\right)$$

for the subgroup of those matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in the modular group such that the equation

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

admits a solution

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

in the modular group. The Hecke subgroup of the modular group associated with a positive integer r is obtained when

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$$

The subgroup is the set elements of the modular group whose lower left entry is divisible by r.

The Hecke subgroup has finite index in the modular group. The Hecke operator $\Delta(r)$ takes a modular form $F(z)$ of order ν into a modular form obtained by averaging

$$F(rz)$$

over the action of elements of the modular group which represent the cosets with respect to the Hecke subgroup.

Hecke operators commute. The identity

$$\Delta(m)\Delta(n) = \sum \Delta(mn/k^2)$$

holds for all positive integers m and n with summation over the common positive divisors k of m and n. Hecke operators apply Fourier analysis in a context without additive structure. Commuting operators are produced in a noncommutative matrix context.

Another hint to the proof of the Riemann hypothesis is found in a construction of Johann Radon (1887–1956). The Radon transformation factors the Fourier transformation for the complex plane as a composition with the Fourier transformation for the real line. The resulting relationship between the Laplace transformation as applied on a line and the Laplace transformation as applied on a plane creates a relationship between the Mellin transformation as it applies on a line and the Mellin transformation as it applies on a plane. A mechanism is created for shifting zeros of zeta functions as required for the Riemann hypothesis. The proof of the Riemann hypothesis applies a previously neglected property of the Radon transformation. The transformation is maximal dissipative in a Hilbert space context.

* * *

The Riemann hypothesis was generally accepted as an underlying goal of research when the author of the present apology decided on a career in mathematical analysis. The choice of the Riemann hypothesis as a research interest was however unusual since the research interests of students are ordinarily determined by their teachers. A doctoral thesis is expected of a student in preparation for a career of research and teaching. A student is not admitted to a doctoral program without demonstrating proficiency in qualifying examinations. When a student is admitted, he is expected to find a member of the faculty who will supervise the writing of the thesis. The faculty advisor is a participant in the choice of research objective and often in its implementation. When the thesis is completed, he is the principal source of information concerning its suitability for publication.

The Riemann hypothesis was difficult as a research objective since it was not proposed by faculty and received no encouragement. Academic freedom permitted research on the Riemann hypothesis without permitting the statement of purpose of the work.

This apology describes exceptional circumstances which made the Riemann hypothesis natural as a research objective. The exceptional individuals who were my father and my mathematical mentor were influenced in similar ways, but with dissimilar effectiveness, by the cultural traditions of the eighteenth century. A war replaced my father by my mathematical mentor for the critical years of my education.

The feudal family de Branges originates in a crusader who died in 1199 leaving a coat of arms for the city which he founded. The shield depicts three swords hanging over three coins, in commemoration of a crusade, and the crown which designates a count. The inscription *Nec vi nec numero* alludes to Jeremiah 4:6, *Not by might, nor by power, but by my Spirit, sayeth the Lord of Hosts.*

The family de Branges is documented in the archives of Franche–Comté, a part of Burgundy when the army of Charles the Bold was defeated at the battle of Morat in 1476. Marie de Bourgogne rescued Burgundy from the army of Louis XI by marriage to the Habsburg Emperor Maximilien I. The wealth of Burgundy permitted the creation of an empire which eventually enclosed France on the north and south as well as in the east. Franche–Comté was included in the Spanish half of the empire on its division in 1556. French armies were never successful in closing this line of communication between the two halves of the empire.

The feudal title of count was created by Charlemagne for the defense of territory, a defense which after the twelfth century required the construction of a stone castle. The count was entitled to tribute from those who received his protection. The castle built by the returning crusader marks the founding of Branges as a city. When the castle was destroyed by the armies of Louis XI, the Comte de Branges lost his function and returned to Saint–Amour, where his family has a distinguished record of service to church and state beginning in the twelfth century.

Saint–Amour lies near the frontier of Franche–Comté with Bresse, which became part of France in 1601 by the Treaty of Lyon. The family Coligny d'Andelot on the French side of the frontier is significant in French history. François d'Andelot and his brother, Admiral Gaspard de Coligny, were leaders of the Protestant Faction in Wars on Religion (1558–1569). Gaspard de Coligny was assassinated in 1572 during the Massacre of Protestants on Saint Bartholomew's Eve.

Although Louis XIV maintained the most powerful army in Europe, he was unsuccessful in extending French territory by force. Franche–Comté was never conquered by France but became part of France in 1678 under the Treaty of Nijmegen. In 1679 François de Branges of Saint–Amour acquired Bourcia, a castle on the Little Mountain which defends Franche–Comté on its frontier with Bresse. Joachim Guyénard, privy counselor to Louis XIV, acquired Andelot, a castle on the other side of the Little Mountain, from the last Coligny owner in 1702.

Andelot and Bourcia lost their purpose for defense when the Little Mountain ceased to be an international frontier. An economic imbalance was created when the owners of castles continued to receive revenue for a function which they no longer performed. Although reasonable proposals were presented at the French court to restore equilibrium, Louis XVI was unable to implement them. The stalemate was a cause of the French Revolution. Bourcia was destroyed in 1792 during a Reign of Terror which followed the revolution. Although Andelot was only lightly damaged, the last male Guyénard emigrated to Philadelphia in 1793.

The Enlightenment stimulated mathematical analysis in France during the eighteenth century without producing the political changes required to meet social needs. The persecution of Protestants caused an emigration of talent from France comparable to the emigration of talent from Germany resulting from the persecution of Jews in the twentieth century. European countries which benefited from French culture looked favorably on political evolution in France until the French Revolution demonstrated the dangers of unrestricted freedom. When the Declaration of Independence was signed in 1776, Philadelphia attracted those who sought peaceful democratic change. The political economist Pierre Samuel du Pont de Nemours emigrated to Philadelphia during the Reign of Terror. Together with his son Eleuthère Irénée du Pont de Nemours he founded a powder mill on the Brandywine near Philadelphia which is the origin of the present Du Pont Company.

The Comte de Branges de Bourcia returned to Franche–Comté after escaping to Geneva with his family during the Reign of Terror. The effect of the revolution was a loss of property but not a loss of life. Octave, the younger of two sons, was born 1825 in Poligny.

The consequences of the revolution were less drastic in overseas territories. The abolition of slavery declared in the revolution was not effective in Guadeloupe where it seemed essential to the harvesting of sugar. When the abolition of slavery was implemented in the Revolution of 1848, former slaves continued to harvest sugar as agricultural workers paid minimal wages. Although sugar continued to be a profitable crop, some plantation owners returned to France where they planted vineyards near Bordeaux.

The effect on the United States of political activity in France was to restrain political change to a level which could be maintained. The abolition of slavery was delayed until it could be implemented without destruction of the republic. Major political change, when it eventually came, was as destructive of lives and property in the United States as it was in France. Political change alone fails to solve economic problems.

The parallel evolution of democratic government in France and the United States created admirers of France in the United States as it created admirers of the United States in France. An admirer in the United States was Augustus Atlee, first Judge of the Supreme Court of Pennsylvania. Judge Atlee worked at the same table in Independence Hall where the Declaration of Independence was signed. The body of law which he wrote for the State of Pennsylvania is accepted in other states of the Union, including the State of Indiana where the proof of the Riemann hypothesis is written.

Sympathy with France helped Franklin Atlee, grandson of Augustus Atlee, make the choice of Paris for a medical education. The choice was not difficult since Paris had an international reputation for medical research. In 1850 he married Louise Caussade, granddaughter of Maximilien de Vernou–Bonneuil through his second child Victoire. The Caussades and de Vernou–Bonneuils were owners of sugar plantations who returned to France from Guadeloupe. Maximilien retained the title of marquis as did others whose hereditary rights were abrogated by revolution.

Dr. Atlee and his bride made their home at Ford–Hook, the Atlee estate in Wayne. His successful career is due not only to advanced medical knowledge but also to the remarkable energy which he applied to his work even in old age. He frequently treated patients in Lancaster County, Pennsylvania, whose religious beliefs were at odds with modern medical practice. When performing a Caesarean operation at an Amish farmhouse, he was surrounded by bearded neighbors arriving in horse–drawn wagons with the intention to lynch him. In reply he asked them to wait until the baby is born.

Félicie de Vernou–Bonneuil, eighth child of Maximilien, married Octave de Branges de

Bourcia. Relevant to this apology are their children Louis, born in 1875, and Paul, born in 1877, in Neuilly–sur–Seine. In 1892 daughter Adèle of Franklin Atlee made her first visit to her grandmother, Victoire Caussade and her aunt Félicie Caussade, Vicomtesse de la Jaille, at the Château de la Fautraise, Argenton–Notre–Dame, Mayenne. Adèle Atlee was an actress who performed in classical drama at the Chestnut Street Theater near Independence Hall. She crossed the Atlantic with her mother from New York to le Havre on steamships of the Compagnie Générale Transatlantique. In 1900 she married Louis de Branges de Bourcia.

Louis and Adèle made their home in Bel–Abri, a house which they built on the Atlee estate in Wayne. Louis was well received in Philadelphia. Evidence of acceptance is his promotion as an officer of the Girard Trust Company, the investment bank of Philadelphia. Of their four children, the one male child was named Louis. On his death in 1910 Franklin Atlee left the bulk of his property to his daughter Adèle in preference to his son John and his daughter Marie.

When France declared war on Germany in 1914, a Vicomte de Branges de Bourcia could not refuse service to his country. Louis enlisted as a private but was promoted to sergeant for bravery in action. Although he was discharged for medical reasons in 1916, he never recovered from gas and shell–shock. He died after the Armistice and is buried in Paris.

Adèle survived her husband by four years. The Atlee estate was sold and the inheritance was divided between three surviving children, of which only her son Lou is relevant to the present apology. At the age of twenty–five Lou sailed to France never to return to the United States. He was received in Paris by his uncle Paul and by Granger descendants of the fourth child, Laure, of Maximilien de Vernou–Bonneuil. The Grangers were also plantation owners from Guadeloupe who returned to France. Family connections have an importance in France which they do not have in the United States. Lou obtained a position in the Compagnie Générale Transatlantique, whose office was in the Grand Hôtel, Place de l'Opéra, as it had been in 1892 when his mother first came to France. Lou supplemented a dollar income by selling transatlantic passages on the de Grasse, the Champlain, the Normandie, and the Ile–de–France.

The dollar was strong against other currencies during the Great Depression. Before leaving for France, Lou married Diane Mc Donald, whose knowledge of French was acquired at the Agnes Irwin School, Bryn Mawr. Wayne and Bryn Mawr are adjacent suburbs of Philadelphia. Her fluency in French was tested by the exclusive use of that language with her husband. They had three children, Louis born in 1932 in Neuilly–sur–Seine, Elise born in 1935, and Eléonore born in 1938. The first of these, Lilou, is the author of the present apology.

My mathematical mentor entered my life at such an early time that memory fails to record it. I opened the door of my grandparents' home in Swarthmore to him and to his wife when I was two years old. Swarthmore is a suburb of Philadelphia comparable to Wayne and Bryn Mawr. The meeting is remembered by my mother because I was naked.

Irenée du Pont was an executive officer of the du Pont Company, later its president and on retirement a member of the board, who was instrumental in the transition from a national producer of gunpowder to an international chemical giant. Nothing in his behavior betrayed the responsibilities he carried as the company met the challenges of depression and war. Although he spoke with light humor, there was an underlying seriousness and content to what he said. His face was tanned as if he had just returned from vacation. He met the heat of summer by drinking orange juice flavored with rhum and by puffing a curved mahogany pipe. As a display of nonchalance he would blow smoke rings. He escaped the cold of winter by going to Xanadu, his vacation home on Varadeiro Beach, Cuba.

Irene du Pont had a face wrinkled like a frog. I have seldom met anyone of more benevolent character. Irene du Pont and her brother Felix du Pont responded to the Great Depression by magnanimous contributions to charitable purposes. She participated as donor when her brother founded Saint Andrew's School, an Episcopal boarding school in Middletown, Delaware. And she supported the Episcopal Diocese of Delaware by donations to the cathedral in Wilmington.

The lives of Irénée and Irene du Pont emulate in an American context the lives of their French ancestors. An anology can be found between Granogue, their home near Wilmington, and a French château. The relationship is not shown in external appearance but in the functional conception of both residences. In both cases the construction is subordinated to the natural setting in which it is made. Children are raised in an environment which preserves family tradition. Granogue lies on a hill which descends to the Brandywine, the river on which Eleuthère Irénée du Pont de Nemours built the first powder mill. His lifesize portrait was displayed in the Granogue living room.

The lives of Irénée and Irene du Pont emulate the lives of their French ancestors in another way since they had as many children as Maximilien de Vernou–Bonneuil. All but two of them were girls. Since one boy and one girl died before I was born, the one surviving boy was left with seven older sisters.

Irene du Pont enlisted the assistance of a schoolmate, Rebecca Motte Frost, for raising her daughters. Since more than a century has passed since the Civil War, some Americans have forgotten the devastation and poverty caused for those who lost the war. Aunt Reba was the daughter of a Confederate general from Charleston, South Carolina. The daughters learned the best traditions of a vanished aristocracy. The relationship between Irene du Pont and Aunt Reba seldom revealed the difference in fortune between those on the winning side and those on the losing side of the war.

Irénée du Pont was, like Alexander von Humboldt, an amateur scientist who expended a personal fortune on his research interests. Like Humboldt he had a professional as a close friend in his scientific exploration. That scientist was my maternal grandfather, Ellice Mc Donald. My grandfather was an associate professor at the University of Pennsylvania when he began his association with Mr. du Pont. My grandfather acquired a reputation as a surgeon by removing cancers which other surgeons considered hopeless. He developed a research interest in the causes of cancer and its treatment by nonsurgical means. He became Founder and Director of the Biochemical Research Foundation of the Franklin Institute, a research center funded by Mr. du Pont. In 1941 the research center moved from its initial building in Philadelphia to a new building adjacent to the campus of the University of Delaware. A cyclotron was built which was later used under the Manhattan Project to measure neutron effects on animals.

My grandfather served Mr. du Pont in the same way that his father served the Hudson's Bay Company as a Chief Factor in Alberta. When Sitting Bull crossed the border into Canada after annihilating the Custer Cavalry in the Little Big Horn, the Chief Factor rode into the Indian Camp with other precursors of the Royal Canadian Mounted Police to explain the Queen's Law in Canada. In so doing they pushed aside drunken Indians celebrating their booty in scalps and horses. My grandfather served Mr. du Pont in the same way that his father served the Queen and that his brother served the Queen as a Canadian general in the First World War.

My grandfather never submitted research proposals to Mr. du Pont. Instead of requesting funds, my grandfather described recent advances in research, not restricting himself to those made by his own organization. He met Mr. du Pont informally at Granogue.

Discussions continued at the Concord Country Club after a round of golf on Sunday morning. Sometimes Mr. du Pont arrived in a small black Cadillac at the house which my grandfather built near Wilmington when the Biochemical Research Foundation moved to Delaware. My grandparents accompanied Mr. and Mrs. du Pont when they took their winter vacation in Cuba. Discussion of research continued on Varadeiro Beach. Mr. du Pont awarded funds to research which attracted his interest.

A privilege awarded to my father by the Compagnie Générale Transatlantique was a yearly round–trip passage to New York. My father never availed himself of this privilege but let it be applied to my mother and to children as they appeared. A Vicomtesse de Branges de Bourcia travels first–class. When my mother returned to France in 1933, she happened to fall into conversation with Pierre du Pont, the older brother of Irénée du Pont who preceded him as president of the du Pont Company. Mr. du Pont described the purchase of Andelot. My information is taken from a letter subsequently written by my father to my grandmother. According to the archives of Franche–Comté, Andelot was bought in 1926 by Lammot Belin. Alice Belin is the wife of Pierre du Pont. Alice and her brother Lammot Belin are descended in a maternal line from the Guyénard who emigrated to Philadelphia.

The outbreak of war had a significance for my family which my mother describes with accurate detail in her *Wartime Experiences, 1939–1941.* This memoir permits me to recall those events, confirming her account. A fundamental change occurred in her life and mine. By writing this memoir she underwent a therapy which permitted the acceptance of diminished circumstances.

Diane Mc Donald, Vicomtesse de Branges de Bourcia, at age sixty.

My father enlisted in the French Army when France declared war and was assigned as a liason officer with the Grenadier Guards. He was evacuated from Dunkirk after the German breakthrough in the spring of 1940.

I completed a second year of schooling in Louveciennes during the first winter of the war. When spring arrived, I left with my mother and sisters for a summer vacation at the Château de la Fautraise. The current Vicomtesse de la Jaille lost her husband in the First World War. Her daughters Audette and Simone were comparable in age and appearance to my mother. My mother was given a room, suitable for a museum, with canopied bed, which had once been used by Louis–Philippe, king of France after the revolution of 1848.

Château de la Fautraise, pen and ink drawing, signed Diane de Branges de Bourcia, June, 1940.

Our hostess responded to distressing war news by playing Chopin on her piano. The Château de la Fautraise filled with refugees as war progressed. The Germany Army arrived as my sister Elise was shrieking protests at my threats to push her into a pond. An officer in a staff car, accompanied by a truckload of soldiers, inspected our château before deciding on another for his headquarters. I learned that he was German after Audette had taken me by the hand to feed the ducks.

The armistice which ended hostilities was celebrated in Argenton–Notre–Dame by opening bottles of vintage wine. The Château de la Fautraise was emptied of refugees. We returned to Paris on a slow train filled with people and luggage. When we arrived in the morning at the Gare Montparnasse, my mother was unable to order a breakfast of coffee and croissants. When we arrived in Louveciennes, we found our house occupied by German soldiers. It was not returned without breakage and loss.

My father returned to France by ship to Bordeaux, passing on his way the sunken Champlain in the Pointe de la Chèvre, a favorite vacation spot near Cherbourg. The Compagnie

Générale Transatlantique lost its offices in the Grand Hôtel, when its guests became high–ranking German officers. The Compagnie Générale Transatlantique maintained existence as a support group distributing reserves of food to former employees.

My third school year in Louveciennes was successfully completed despite the use of the school building as a German military headquarters. Classes were held in the mairie. I do not recall any hindrance resulting from improvised equipment. Madame Jammet taught me the most important skill which can be acquired in education at any level, which is a love for the material taught. My career as a professor of mathematics has been facilitated by my own ability to teach this skill since the Riemann hypothesis has over fifty years received minimal acceptance as a research aim.

In the spring 1941 my grandfather insisted that his daughter and grandchildren return to Philadelphia. My grandfather expected that my father would return for active duty as second lieutenant in the United States Army Reserve. The decision made by my father to remain in France is not understandable without a knowledge of European history. The arrival of Adolf Hitler to power in Germany created an unprecedented political situation. A Winston Churchill was required to identify a political leader who aimed to reverse centuries of social progress. A Charles de Gaulle was required to unite France into effective resistance. Cooperation between French speaking and English speaking peoples removed an enmity created by centuries of political and economic competition.

My mother took the train to Lisbon with her three children to await passage on a ship bound for New York. Since I was the only male of the remaining family, I acquired responsibilities which later restricted my ability to create a family of my own or to pursue a normal career. Since my next meeting with my father occurred after more than twenty years of postdoctoral research on the Riemann hypothesis, he has no further place in this apology.

The remainder of the war was spent at a summer cottage in Rehoboth Beach, Delaware. The transition to English as a language seems to have stimulated mathematical ability. The first signs were an ability to decode cryptograms in the Philadelphia Enquirer. Words provide a stimulus which escapes the logical function of the mind by penetrating into the subconscious. The logical function is boosted when it benefits from the subconscious function.

The end of childhood was caused by two events when I was twelve. I entered the second form at Saint Andrew's School as the cottage in Rehoboth was sold. My new home was the house near Wilmington which my grandparents were building when I came from France. My grandmother replaced my mother as the central person in family life.

My grandmother was the daughter of a German immigrant who made a successful career in Philadelphia. Philip Hübner was born in Mosbach, a village upstream from Heidelberg on the Neckar. He must have been a student at the University of Heidelberg since he spoke both French and English. His refusal of military service in the Franco–Prussian War is consistent with student character. He fled to England but went to Paris when war ended. An American business man invited him to come to Philadelphia.

His daughter, Ann Heebner, grew up in Chestnut Hill, a suburb of Philadelphia comparable to Wayne. She attended the same art school in Philadelphia as Mary Cassatt. Oil paintings of that time indicate talent. The First World War may be the reason why she did not become a professional artist. Germans in America lost the sympathy which lubricates business relations. After the war she did not have the means to travel to Italy and Spain as Mary Cassatt did in the nineteenth century. She spent a winter in Paris as a student of Whistler before war began.

My grandmother produced good paintings after she married my grandfather. The tulips

Dr. and Mrs. Ellice Mc Donald at Invercoe, the home built near Wilmington in 1941. Photo taken circa 1954.

of her Swarthmore garden are painted with an intensity of color reminiscent of Renoir. A vivid self–portrait competes with the best I have seen. But her energies were given to support of my grandfather in his work. She was an excellent cook and gardener. Her friendship with Irene du Pont and Aunt Reba complemented the friendship of my grandfather with Irénée du Pont.

The challenges of war and the arrival of my mother in America tested the best qualities of my grandmother. Inflation caused a reduction in purchasing power not compensated by increases in salary. My mother caused a reduction of disposable income. My grandmother decided to do her own cooking and sent her maid to help my mother in Rehoboth. Agnes was a young woman from the Appalachians who was loved by my sisters. She returned home to be married when Nora no longer required her attention.

My mother and sisters were received by the mother of a classmate, Mary Read, at the Agnes Irwin School. Mrs. Read and her sister Miss Marion Wood owned adjacent estates in Bryn Mawr on a hill which descended through woods to the Schuylkill River. Across the river the Alan Wood Steel Company supplied work to the Poles, Ukrainians, Hungarians, Chechs, and Slovaks who lived in the colorful houses of Conshohocken. The discordant sounds of the factory were carried by the wind to the hilltop. Mrs. Read and her sister were Quakers who gave a high priority in their lives to charitable actions. My mother was treated as if she were a daughter.

Years were required for my mother to free herself from dependence on parents. Awareness of the burden on my grandmother made me treat my studies at Saint Andrew's School seriously. I was disappointed by my performance in algebra. In a summer vacation I solved several hundred problems in an exercise book on algebra. The experience was instructive because I had no text explaining how to solve problems. I performed sufficiently well in a proficiency examination to be advanced to plane geometry when I returned to school.

Saint Andrew's School offered a study incentive of weekends off campus to students who obtained satisfactory grades. I used this means to visit my grandparents in Wilmington. I would then caddy for my grandfather as he played golf on Sunday morning with Mr. du Pont. One morning, as Mr. du Pont was drinking his usual glass of rhum and orange juice after the game, he asked me to find positive integers a, b, and c such that

$$a^3 + b^3 = 22c^3.$$

There was no explanation that this was a variation due to Lagrange of a problem of Fermat. Nor was any hint given as to how the problem should be approached.

The solution of the problem required more than a year. In the process I learned the available sources of information about mathematics. Some discoveries were surprising. On summer days my sister Elise liked to play the piano in the Read home at the top of the hill. Her favorite piece by Beethoven is dedicated to a woman of like name. As she played, I read well written and historically interesting articles on mathematics in the Encyclopedia Britannica. There was enough detail given for me to reconstruct the representation of positive integers in the form

$$a^2 + b^2.$$

I then constructed for myself the representation of positive integers in the form

$$a^2 - ab + b^2$$

as required for the solution of the Lagrange problem.

I do not remember receiving approval for my solution of the Lagrange problem. In solving the problem I became effectively a member of the Biochemical Research Foundation. The challenges of Saint Andrew's School were secondary to life with my grandfather. The Lagrange problem taught me to work without expecting reward and yet believing that I would benefit from the work done. The Lagrange problem also taught me to search for mathematical information from nonmathematical sources.

Mathematicians, like Mr. du Pont, do not like to commit themselves to a meaning for their actions since any such explanation is necessarily subjective and open to criticism. Their intentions are concealed in precise detail which exhausts the patience of a reader. Explanations of meaning require a decision as to the purpose of mathematics. I learned the Euler theory of the gamma function and the Mark Krein theory of the vibrating string from *The Mathematics of Physics and Chemistry* by Henry Margenau and George Murphy. Since the proof of the Riemann hypothesis is a continuation of both theories, experience at the Biochemical Research Foundation prepared the proof of the Riemann hypothesis.

In school years I benefited from the good will of friends of the Atlee and de Branges families in Wayne. The principal one for me (since others benefitted my sisters) was Colonel Harrison Smith, a lawyer and reserve army officer who served in the American Expeditionary Force during the First World War. A lover of nature, he was my father's scoutmaster in Wayne. It disturbed Colonel Smith, as it disturbed Mrs. du Pont, that sons should grow up without a father. He supported Camp Pasquaney in the White Mountains of New Hampshire for the same reasons that Mrs. du Pont supported Saint Andrew's School. The two summers I spent at Camp Pasquaney were at his expense. If the five years I spent at Saint Andrew's School were at the expense of Mrs. du Pont, the secret was buried with my grandmother.

Camp Pasquaney and Saint Andrew's School made contributions to my mathematical career. In school I learned the Christian concepts of faith and of the forgiveness of sins. For mathematicians an error is a sin. I do not expect from myself a performance which is free of error any more than I would of another sin. I am unlike a majority of mathematicians in believing that the sin of error can be forgiven. I acknowledge my errors and learn from them. I have seldom made an error which does not contain some grain of truth. Faith is the capacity to maintain the search for truth despite repeated error.

In camp I learned the pleasure of hiking through mountainous terrain, a pleasure heightened by gifts of bittersweet chocolate from Colonel Smith. These are conditions which are known to stimulate thought. For me the stimulus lies in the changing perspectives of surfaces which are not flat. All major mathematical contributions of my career have been made in summer months when hiking was possible.

Two events prepared the future as I approached graduation from Saint Andrew's School. My grandfather decided that I should have a university education. In retrospect it seems to me that Mr. du Pont made the decision since my grandfather had obligations which took precedence over my education. The choice of the Massachusetts Institute of Technology is consistent with a decision by Mr. du Pont since it was his university. I had scores on entrance examinations which cleared obstacles to admission. My grandfather and I flew to Boston for a speech given by Winston Churchill. Our holiday mood is preserved in my memory by its association with the melody *Greensleeves* played in our Brookline hotel. Then we crossed the Charles to examine the campus which was to give me four determining years of education.

The other event concerned the property which my father inherited from his mother. The income from these stocks and bonds became available to my mother when she returned to the United States in 1941. The provision remained in force in the expectation that my father would return. It became clear that my father had no intention of returning. Colonel Smith, with information from Colonel Lyle on active duty in Germany, concluded that my father had deserted wife and children by not returning to the United States. He represented her in a divorce suit which awarded her possession of my father's inherited property. Inflation left a diminished sum which was nevertheless sufficient to buy a house in Wayne. Our family was united for years in which my mother and sisters required my presence.

When I arrived in Boston in September 1949, I was asked to join the fraternity which had been that of Mr. du Pont, but declined in favor of the Boston branch of the fraternity which had been that of my father at the University of Pennsylvania. The Number Six Club, originally located at number six Louisburg Square, was then an international center patterned on a Harvard club. Its members were favored guests at parties where formal dress was worn in an aura of New Orleans jazz.

I treated my undergraduate studies as if I were a graduate student. George Thomas was writing a text on the calculus and analytic geometry which was tested on the incoming freshman class. Professor Thomas himself taught the section in which I was placed. I worked through the exercises for all four semesters and was exempted from the remaining three semesters by a proficiency examination. Professor Thomas was pleased by my reading of his untested lecture notes. I remember that he attended a wrestling match against Boston University in which I pinned an opponent in a weight class above my own 155 pounds.

I was freed in the second semester to take a graduate course in linear analysis taught by Witold Hurewicz. The text by Garrett Birkhoff and Saunders Mac Lane was familiar as it had been in the library of the Biochemical Research Foundation. It must have been ordered for my benefit as no other reader is conceivable.

In the summer break I read the recently published *Lectures on Classical Differential Geometry* by Dirk Struik. When my knowledge was tested in a proficiency examination, Professor Struik gave me more than a perfect score since I had to correct the statement of one of the problems before solving it. The differential geometry of Cartesian space is a continuation of Newtonian analysis made in the eighteenth century. Professor Struik notes that mathematical contributors made contributions to the political changes which culminate in the French Revolution. His interest in politics was sufficient for him to be investigated as a communist by the un–American Activities Committee of the Senate. In an extended sabbatical leave he wrote *A Concise History of Mathematics*, which attempts a coherent view of mathematical analysis.

In my sophomore year I took a course from Walter Rudin on the *Principles of Mathematical Analysis.* His exemplary teaching was made possible by a C.L.E. Moore Instructorship. For the first time I took careful lecture notes as later in graduate courses. *The Theory of Functions* by Edward Titchmarsh however treats the material in a more stimulating way. The Riemann hypothesis is an underlying theme for Titchmarsh as well as for Taylor Whittaker and Neville Watson in their classical text *Modern Analysis.*

The decision to prove the Riemann hypothesis begins as a search for purpose in my junior year. Substantial mathematical sources were available at a time when the quality of books was not measured by circulation. Books on Diophantine analysis were obtainable in the library of Saint Andrew's School, in the Wilmington Public Library, and in the library of the University of Delaware. The applications to Diophantine analysis of the structure of algebras were available in the University of Chicago monograph by Eugene Dickson on *Algebras and their Arithmetics.* When I read the lecture notes of George Thomas, I was aware that he was applying the Hamiltonian theory of quaternions. These hypercomplex numbers, which in Diophantine analysis are applied to the representations of a positive integer as a sum of four squares, clarify the Newtonian analysis of Cartesian space. Their application in electromagnetism is a challenge to those who have an interest in Newtonian analysis. In an undergraduate thesis I proposed an interpretation of electric and magnetic vectors as Christoffel symbols of a Riemannian manifold.

A continuation was made in graduate years at Cornell University, where I obtained a teaching assistantship on the recommendation of George Thomas. I approached graduate studies as if I were a postdoctoral fellow. My grandfather died in January of my second year as I was taking doctoral qualifying examinations. The lecture notes of Emil Artin and Amelia Noether, taken by Bartel van der Waerden and published by Springer–Verlag as *Moderne Algebra*, were stimulating preparation for the examinations. I was impressed by the relationship between convexity and topology in the geometric formulation of the Hahn–Banach theorem, due to Marshall Stone and presented by Jean Dieudonné under the pseudonym Nicolas Bourbaki as *Espaces Vectoriels Topologiques.* My teaching duties were reduced by a fellowship in the academic year following the qualifying examinations.

The Riemann hypothesis emerged as a research objective because of its link through Fourier analysis to quantum mechanics. The *Introduction to the Theory of Fourier Integrals* by Edward Titchmarsh led me into *Eigenfunction Expansions Associated with Second–Order Differential Equations* and *The Theory of the Riemann Zeta–Function.* An opening for research appeared in a symposium on harmonic analysis held at Cornell University in the summer 1956.

Mathematical symposia are ordinarily quiet events whose dullness is compensated by

Louis and Elise de Branges de Bourcia, Prince Akihito, and another guest of Elizabeth Gray Vining, author of *Windows for the Crown Prince*, for breakfast at her home in Mount Airy, Philadelphia. Prince Akihito brought his English teacher a miniature replica of a golden chariot on returning to Japan from a European tour. The mothers of Elizabeth Gray Vining and Ann Heeber Mc Donald were sisters. Their common grandmother was a descendant of a Quaker immigrant who obtained land from William Penn across the Delaware River from Philadelphia.

the beauty of the places where they are held. Szolem Mandelbrojt of the Hebrew University Jerusalem gave a polished lecture on the Carleman method in the spectral theory of unbounded functions. The lecture became interesting after the furious rebuttal given by Arne Beurling of the Institute for Advanced Study. The Carleman method applies two functions, one analytic above the real axis and another analytic below the real axis, whereas the Wiener operational calculus requires a function defined on the real axis. A clarification is needed of the auxiliary functions introduced by Carleman.

An explanation is given in my doctoral thesis, written in the academic year 1956–1957. I would not have accepted the problem without the encouragement of Harry Pollard, who organized the symposium, and of Wolfgang Fuchs, who guided me through literature in complex analysis. The monograph by Ralph Boas on *Entire Functions* prepared me in a mathematical specialty which includes the classical approach to the Riemann hypothesis. My thesis, *Local operators on Fourier transforms*, clarifies the appearance of entire functions in the spectral theory of unbounded functions.

My last visit to Mr. du Pont was made with my grandmother and my sister Nora when I had become a doctor of philosophy and had completed active duty as a lieutenant in the United States Army Reserve. Daughters arrived to see their mother, in bed upstairs with a terminal illness. Mr. du Pont, blinded my cataracts in his eyes, drank rhum and orange juice as he joked about the revolution in Cuba. Xanadu was converted by Fidel Castro into a museum of capitalist decadence with the former butler as curator. The amusement shown by Mr. du Pont at my mathematical career is a rare hint of a secret subsidy of my education in Boston.

* * *

The issue in the Riemann hypothesis is the extension of a function, which is analytic and without zeros in a half–plane, to a function which is analytic and without zeros in a larger half–plane.

An analytic weight function is a function which is analytic and without zeros in the upper half–plane. The weighted Hardy space associated with an analytic weight function $W(z)$ is the set $\mathcal{F}(W)$ of functions $F(z)$, analytic in the upper half–plane, such that the least upper bound

$$\|F\|^2_{\mathcal{F}(W)} = \sup \int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 dx$$

over all positive numbers y is finite. The weighted Hardy space is a Hilbert space which contains the function

$$W(z)W(w)^-/[2\pi i(w^- - z)]$$

as reproducing kernel function for function values at w when w is in the upper half–plane. Multiplication by

$$(z-w)/(z-w^-)$$

is an isometric transformation of the space onto the subspace of functions which vanish at w when w is in the upper half–plane.

A sufficient condition for analytic continuation without zeros of the analytic weight function $W(z)$ to the half–plane

$$-1 < iz^- - iz$$

is given by a hypothesis on the weighted Hardy space $\mathcal{F}(W)$. The condition states that a maximal dissipative relation is defined in the space by taking $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space.

An analytic weight function $W(z)$ is said to be an Euler weight function if a maximal dissipative relation is defined in the weighted Hardy space $\mathcal{F}(W)$ for all h in the interval $[0,1]$ by taking $F(z)$ into $F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space.

If a maximal dissipative relation in the weighted Hardy space $\mathcal{F}(W)$ is defined by taking $F(z)$ into $F(z+i)$ whenever $F(z)$ and $F(z+i)$ belong to the space, then

$$W(z) = U(z)V(z)$$

is the product of an Euler weight function $V(z)$ and an entire function $U(z)$ without zeros which is periodic of period i.

An example

$$W(z) = \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz)$$

of an Euler weight function, ν a nonnegative integer, appears in Fourier analysis on Cartesian space. Since Cartesian space is deficient in multiplicative structure, a reformulation in Fourier analysis on quaternions is indicated. Quaternionic space is treated as a covering space of Cartesian space whose covering sheets are parametrized by real numbers. The self–conjugate component of a quaternion

$$t + ix + jy + kz$$

is treated as time whereas the skew–conjugate component is treated as an element of Cartesian space.

The quantum mechanical formulation of motion of a particle in Cartesian space applies the Fourier transformation to relate square integrable functions of position to square integrable functions of momentum. When the particle has maximal symmetry, these functions of position and momentum are acted on by the compact group of quaternions whose conjugate is its inverse. The group is a double covering of the group of rotations of Cartesian space.

The Hilbert space of square integrable functions on quaternionic space decomposes into invariant subspaces, parametrized by nonnegative integers ν, under the action of the group. Functions of order ν are mapped into functions of order ν by the Fourier transformation.

The Laplace transformation has a relationship to the Fourier transformation which was applied by Fourier to the flow of heat. The Laplace transformation is advantageous for the computation of Fourier transforms when symmetry is present, as it is for functions of order ν.

The Mellin transformation further simplifies the computation of Fourier transforms in the presence of symmetry. The relationship of the Mellin transformation to the Laplace transformation produces weighted Hardy spaces. The Euler weight function

$$W(z) = \Gamma(\tfrac{1}{2}\nu + \tfrac{1}{2} - iz)$$

applies to Mellin transforms of functions of order ν.

Experience with the real line indicates how constructions in Fourier analysis can be adapted to new topologies so as to produce zeta functions. The analogue for quaternionic space of the rational numbers are the quaternions with rational coordinates. Since the ring of integral quaternions admits an Euclidean algorithm, a nontrivial ideal is generated by an element of the ideal. Compactifications of the quaternionic space are constructed by a mixing of the topology of the quaternionic space with the topology of an adic completion. Euler weight functions are obtained which are the products of two factors, one of which is constructed from the gamma function and the other from a Hecke zeta function.

An Euler weight function contains complete information concerning the context of Fourier analysis in which it arises. For this purpose Hilbert spaces, whose elements are entire functions, are considered which have these properties:

(H1) Whenever $F(z)$ is in the space and has a nonreal zero w, the function

$$F(z)(z - w^-)/(z - w)$$

belongs to the space and has the same norm as $F(z)$.

(H2) A continuous linear functional on the space is defined by taking $F(z)$ into $F(w)$ for every nonreal number w.

(H3) The function

$$F^*(z) = F(z^-)^-$$

belongs to the space whenever $F(z)$ belongs to the space, and it always has the same norm as $F(z)$.

If an entire function $E(z)$ satisfies the inequality

$$|E^*(z)| < |E(z)|$$

when z is in the upper half–plane, then it has no zeros in the upper half–plane. A weighted Hardy space $\mathcal{F}(E)$ exists. The set of entire functions $F(z)$ such that $F(z)$ and $F^*(z)$ belong to the space $\mathcal{F}(E)$ is a Hilbert space $\mathcal{H}(E)$ which is contained isometrically in the space

$\mathcal{F}(E)$ and which satisfies the axioms (H1), (H2), and (H3). The space $\mathcal{H}(E)$ contains the entire function

$$\frac{E(z)E(w)^- - E^*(z)E(w^-)}{2\pi i(w^- - z)}$$

of z as reproducing kernel function for function values at w for all complex numbers w. A Hilbert space, whose elements are entire functions, which satisfies the axioms (H1), (H2), and (H3), and which contains a nonzero element, is isometrically equal to a space $\mathcal{H}(E)$.

A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is said to be an Euler space of entire functions if for h in the interval $[0, 1]$ a maximal dissipative transformation in the space is defined by taking $F(z)$ into $F(z + ih)$ whenever $F(z)$ and $F(z + ih)$ belong to the space. The defining function of a space $\mathcal{H}(E)$ which is an Euler space of entire functions admits no distinct zeros w^- and $w - ih$ with h in the interval $[0, 1]$.

If $W(z)$ is an Euler weight function, the set of entire functions $F(z)$ such that $F(z)$ and $F^*(z)$ belong to the weighted Hardy space $\mathcal{F}(W)$ is an Euler space of entire functions which is contained isometrically in the space $\mathcal{F}(W)$ and which contains a nonzero element.

There are many Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3) and which are contained isometrically in the space $\mathcal{F}(W)$. Interesting spaces have the property that an entire function $F(z)$ belongs to the space if $(z-w)F(z)$ belongs to the space for some complex number w and if $F(z)$ belongs to the space $\mathcal{F}(W)$. A fundamental theorem states that such Hilbert spaces of entire functions are totally ordered by inclusion. All such spaces are Euler spaces of entire functions. Many such spaces exist in the sense that the elements of the weighted Hardy space are recovered by an expansion similar to a Fourier expansion.

The Riemann hypothesis is proved for a Dirichlet zeta function by constructing a Hecke zeta function whose zeros contain the zeros of the Dirichlet zeta function. The same construction is applied in the proof of the Riemann hypothesis for the Euler zeta function. A modification is required since an Euler weight function cannot have a singularity at the origin. The weighted Hardy space obtained fails to have the required maximal dissipative properties. The transformation which takes $F(z)$ into $F(z+ih)$ whenever $F(z)$ and $F(z+ih)$ belong to the space is dissipative when h belongs to the interval $[0, 1]$. When h is positive, a one–dimensional extension exists which is a maximal dissipative relation. This deficiency of the space is compensated by an isometric conjugation which takes $F(z)$ into $F^*(-z)$ whenever $F(z)$ belongs to the space. The analytic weight function has an analytic extension without zeros to a region containing all nonzero elements of the half–plane

$$0 < iz^- - iz.$$